

Rational solutions of the Sasano systems of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$

By Kazuhide Matsuda

Department of Engineering Science, Niihama National College of Technology,
7-1 Yagumo-chou, Niihama, Ehime, 792-8580, Japan.

Abstract

In this paper, we completely classify the rational solutions of the Sasano system of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$ which are all the coupled P_{III} systems and have the affine Weyl group symmetries of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$. The rational solutions are classified as one type by the Bäcklund transformation group.

Introduction

Paul Painlevé and his colleagues [24, 3] intended to find new transcendental functions defined by second order nonlinear differential equations. In general, nonlinear differential equations have moving branch points. If a solution has moving branch points, it is too complicated and is not worth considering. Therefore, they determined second order nonlinear differential equations with rational coefficients which have no moving branch points. As a result, the standard forms of such equations turned out to be given by the following six equations:

$$\begin{aligned}
 P_{\text{I}} &: y'' = 6y^2 + t \\
 P_{\text{II}} &: y'' = 2y^3 + ty + \alpha \\
 P_{\text{III}} &: y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
 P_{\text{IV}} &: y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
 P_{\text{V}} &: y'' = \left(\frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right) + \frac{\gamma}{t}y + \delta \frac{y(y+1)}{y-1} \\
 P_{\text{VI}} &: y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' \\
 &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right),
 \end{aligned}$$

where $' = d/dt$ and $\alpha, \beta, \gamma, \delta$ are all complex parameters.

While generic solutions of the Painlevé equations are “new transcendental functions,” there are special solutions which are expressible in terms of rational, algebraic or classical special functions. Especially, the rational solutions of P_J ($J = \text{II, III, IV, V, VI}$) were classified by Yablonski and Vorobev [34, 33], Gromak [5, 4], Murata [15, 16], Kitaev, Law and McLeod [6], Mazzoco [13], and Yuang and Li [35]. Especially, Murata [15] classified the rational solutions of the second and fourth Painlevé equations by using the Bäcklund transformations, which transform a solution into another solution of the same equation with different parameters.

P_J ($J = \text{II, III, IV, V, VI}$) have the Bäcklund transformation group. It was shown by Okamoto [20, 21, 22, 23] that the Bäcklund transformation groups of the Painlevé equations except for P_I are isomorphic to the extended affine Weyl groups. For $P_{\text{II}}, P_{\text{III}}, P_{\text{IV}}, P_{\text{V}}$, and P_{VI} , the Bäcklund transformation groups correspond to $A_1^{(1)}$, $A_1^{(1)} \oplus A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, and $D_4^{(1)}$, respectively.

Many fourth order Painlevé type equations have now been found. The examples are the Noumi and Yamada systems, Sasano systems, Fuji and Suzuki systems, etc. Our aim is to classify the rational solutions of all the fourth order Painlevé type equations. For this purpose, we will mainly use the residue calculus of their solutions and Hamiltonians.

Noumi and Yamada [18] discovered the equations of type $A_l^{(1)}$ ($l \geq 2$), whose Bäcklund transformation group is isomorphic to the extended affine Weyl group $\tilde{W}(A_l^{(1)})$. The Noumi and Yamada systems of types $A_2^{(1)}$ and $A_3^{(1)}$ correspond to the fourth and fifth Painlevé equations, respectively. Furthermore, we [7, 8] classified the rational solutions of the Noumi and Yamada systems of types $A_4^{(1)}$ and $A_5^{(1)}$, which are fourth order versions of the fourth and fifth Painlevé equations, respectively.

Sasano [26, 27] obtained the coupled Painlevé III, V and VI systems from a higher dimensional generalization of Okamoto’s space of initial conditions, which have the affine Weyl group symmetries of types $B_4^{(1)}$, $D_5^{(1)}$ and $D_6^{(1)}$, and are called the Sasano systems of types $B_4^{(1)}$, $D_5^{(1)}$ and $D_6^{(1)}$, respectively. Moreover, he [28, 29, 30, 31] obtained the equations of many different affine Weyl group symmetries, which are all called the Sasano systems. Especially, for the Sasano systems of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$, see [28, 27]. We [9, 10, 11, 12] classified the rational solutions of the Sasano systems of types $A_5^{(2)}$, $A_4^{(2)}$, $A_1^{(1)}$, and $D_3^{(2)}$.

Fuji and Suzuki [1, 2] obtained the equation of type $A_5^{(1)}$ from a similarity reduction of the Drinfel’d-Sokolov hierarchy, which is called the Fuji and Suzuki system of type $A_5^{(1)}$. Moreover, the Noumi and Yamada system of type $A_5^{(1)}$ is expected to be obtained from the degeneration of the Fuji and Suzuki system of type $A_5^{(1)}$.

Following Oshima’s [19] classification of the irreducible Fuchsian equation with four accessory parameters, Sakai [25] obtained four “source” equations of all the fourth order Painlevé type equations, that is, the well-known Garnier system with two variables, the

Fuji and Suzuki system of type $A_5^{(1)}$, the Sasano system of type $D_6^{(1)}$, and a new one.

Therefore, the Sasano system of type $D_6^{(1)}$ is obtained from the isomonodromic deformations of the Fuchsian type, which implies that from Miwa's theorem [14], the system has the Painlevé property. Therefore, any other Sasano system, including the Sasano systems of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$, is expected to be obtained by the degeneration from the Sasano system of type $D_6^{(1)}$ and have the Painlevé property.

In this paper, we first classify the rational solutions of the Sasano system of type $B_4^{(1)}$, which is defined by

$$B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4} \begin{cases} tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t, \\ ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1, \\ tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t, \\ tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3, \\ \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = 1. \end{cases}$$

This system of ordinary differential equations is also expressed by the Hamiltonian system:

$$t \frac{dx}{dt} = \frac{\partial H_{B_4^{(1)}}}{\partial y}, \quad t \frac{dy}{dt} = -\frac{\partial H_{B_4^{(1)}}}{\partial x}, \quad t \frac{dz}{dt} = \frac{\partial H_{B_4^{(1)}}}{\partial w}, \quad t \frac{dw}{dt} = -\frac{\partial H_{B_4^{(1)}}}{\partial z},$$

where the Hamiltonian H is given by

$$H_{B_4^{(1)}} = x^2y(y-1) + x\{(1-2\alpha_2-2\alpha_3-2\alpha_4)y - \alpha_1\} + ty \\ + z^2w(w-1) + z\{(1-2\alpha_4)w - \alpha_3\} + tw + 2yz(zw + \alpha_3).$$

Our main theorem is as follows.

Theorem 0.1. *For a rational solution of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, by some Bäcklund transformations, the solution and parameters can be transformed so that*

$x \equiv 0$, $y \equiv 1/2$, $z = 1/\{2\alpha_4\} \cdot t$, $w \equiv 0$ and $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$, respectively.

Moreover, for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution if and only if the parameters satisfy one of the following conditions:

- | | | | |
|-----|--|--|---|
| (1) | $\alpha_0 - \alpha_1 \in \mathbb{Z}$, | $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$, | $\alpha_0 - \alpha_1 \equiv 2\alpha_3 + 2\alpha_4 \pmod{2}$, |
| (2) | $\alpha_0 - \alpha_1 \in \mathbb{Z}$, | $2\alpha_4 \in \mathbb{Z}$, | $\alpha_0 - \alpha_1 \equiv 2\alpha_4 \pmod{2}$, |
| (3) | $\alpha_0 + \alpha_1 \in \mathbb{Z}$, | $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$, | $\alpha_0 + \alpha_1 \equiv 2\alpha_3 + 2\alpha_4 \pmod{2}$, |
| (4) | $\alpha_0 + \alpha_1 \in \mathbb{Z}$, | $2\alpha_4 \in \mathbb{Z}$, | $\alpha_0 + \alpha_1 \equiv 2\alpha_4 \pmod{2}$, |
| (5) | $\alpha_0 - \alpha_1 \in \mathbb{Z}$, | $\alpha_0 + \alpha_1 \in \mathbb{Z}$, | $\alpha_0 - \alpha_1 \not\equiv \alpha_0 + \alpha_1 \pmod{2}$, |
| (6) | $2\alpha_3 \in \mathbb{Z}$, | $2\alpha_4 \in \mathbb{Z}$, | $2\alpha_3 \equiv 1 \pmod{2}$. |

In order to explain the method to prove our main theorem, let us define the coefficients of the Laurent series of (x, y, z, w) at $t = \infty, 0, c \in \mathbb{C}^*$ by

$$\begin{cases} a_{\infty,k}, a_{0,k}, a_{c,k} \ (k \in \mathbb{Z}), b_{\infty,k}, b_{0,k}, b_{c,k} \ (k \in \mathbb{Z}), \\ c_{\infty,k}, c_{0,k}, c_{c,k} \ (k \in \mathbb{Z}), d_{\infty,k}, d_{0,k}, d_{c,k} \ (k \in \mathbb{Z}). \end{cases}$$

For example, if x, y, z, w all have a pole at $t = \infty$, we define

$$\begin{cases} x = a_{\infty,n_0}t^{n_0} + a_{\infty,n_0-1}t^{n_0-1} + \cdots + a_{\infty,0} + a_{\infty,-1}t^{-1} + \cdots, \\ y = b_{\infty,n_1}t^{n_1} + b_{\infty,n_1-1}t^{n_1-1} + \cdots + b_{\infty,0} + b_{\infty,-1}t^{-1} + \cdots, \\ z = c_{\infty,n_2}t^{n_2} + c_{\infty,n_2-1}t^{n_2-1} + \cdots + c_{\infty,0} + c_{\infty,-1}t^{-1} + \cdots, \\ w = d_{\infty,n_3}t^{n_3} + d_{\infty,n_3-1}t^{n_3-1} + \cdots + d_{\infty,0} + d_{\infty,-1}t^{-1} + \cdots, \end{cases}$$

where n_0, n_1, n_2, n_3 are all positive integers and $a_{\infty,n_0}b_{\infty,n_1}c_{\infty,n_2}d_{\infty,n_3} \neq 0$. Moreover, we define the coefficients of the Laurent series of $H_{B_4^{(1)}}$ at $t = \infty, 0, c \in \mathbb{C}^*$ by $h_{\infty,k}, h_{0,k}, h_{c,k} \ (k \in \mathbb{Z})$, respectively.

This paper is organized as follows. In Section 1, for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, we determine the meromorphic solutions near $t = \infty$ and prove that for a meromorphic solution near $t = \infty$, z has a pole of order $n \ (n \geq 1)$ at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Furthermore, we find that $a_{\infty,0} = \alpha_0 - \alpha_1$.

In Section 2, for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, we deal with the meromorphic solutions near $t = 0$ and find that $a_{0,0} = 0, \alpha_0 - \alpha_1$.

In Section 3, for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, we treat the meromorphic solutions near $t = c \in \mathbb{C}^*$ and show that for a rational solution of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $a_{\infty,0} - a_{0,0} \in \mathbb{Z}$.

In Section 4, for a meromorphic solution of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ near $t = \infty, 0, c \in \mathbb{C}^*$, we study the Hamiltonian $H_{B_4^{(1)}}$ and compute $h_{\infty,0}, h_{0,0}, h_{c,-1}$, where $h_{\infty,0}, h_{0,0}$ are both expressed by the parameters and $h_{c,-1}$ is $nc \ (n \in \mathbb{Z})$. Therefore, we find that for a rational solution of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, the parameters satisfy $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$.

In Section 5, we define the Bäcklund transformations, $s_0, s_1, s_2, s_3, s_4, \pi_1, \pi_2$ and investigate their properties.

In Section 6, we treat the infinite solutions, that is, solutions such that some of x, y, z, w are identically equal to ∞ .

In Section 7, we treat a rational solution such that z has a pole of order one at $t = \infty$ and obtain the necessary conditions for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ to have such rational solutions. For this purpose, we use the formulas, $a_{\infty,0} - a_{0,0} \in \mathbb{Z}$ and $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$.

In Section 8, we deal with a rational solution such that z has a pole of order $n \ (n \geq 2)$ at $t = \infty$ and obtain necessary conditions for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ to have such rational solutions. For this purpose, we mainly use the formula, $a_{\infty,0} - a_{0,0} \in \mathbb{Z}$.

In Section 9, we summarize the necessary conditions in Sections 7 and 8 and transform the parameters so that either of the following occurs:

$$\text{I : } \alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 0, \alpha_4 \neq 0, \quad \text{II : } \alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2.$$

In this paper, cases I and II are called the standard forms I and II, respectively. For the standard form I, we determine the rational solution of Corollary 1.23 in Section 1.

In Section 10, we treat the standard form II. We then transform the parameters so that either of the following occurs:

$$(1) \alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 0, \alpha_4 \neq 0, \quad (2) \alpha_0 - \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2.$$

In Section 11, we deal with the case where $\alpha_0 - \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2$. We then transform the parameters so that either of the following occurs:

$$(1) \alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 0, \alpha_4 \neq 0, \quad (2) \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1/2.$$

In Section 12, we prove that for $B_4^{(1)}(0, 0, 0, 0, 1/2)$, there exists no rational solution. In Section 13, we prove our main theorem.

In the appendix, following Sasano [28], we introduce the Sasano systems of types $D_4^{(1)}$ and $D_5^{(2)}$ and show that the Sasano systems of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$ are all equivalent by birational transformations. Using Theorem 0.1, we classify the rational solutions of the Sasano systems of types $D_4^{(1)}$ and $D_5^{(2)}$.

1 Meromorphic solutions at $t = \infty$

In this section, we determine the meromorphic solution at $t = \infty$.

1.1 The case where x, y, z, w are all holomorphic at $t = \infty$

In this subsection, we deal with the case where x, y, z, w are all holomorphic at $t = \infty$.

Proposition 1.1. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, y, z, w are all holomorphic at $t = \infty$.*

Proof. Comparing the coefficients of the term t in

$$tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t,$$

we can prove the proposition. □

1.2 The case where one of (x, y, z, w) has a pole at $t = \infty$

In this subsection, we deal with the case in which one of (x, y, z, w) has a pole at $t = \infty$ and consider the following four cases:

- (1) x has a pole at $t = \infty$ and y, z, w are all holomorphic at $t = \infty$,
- (2) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$,
- (3) z has a pole at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$,
- (4) w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$.

1.2.1 The case where x has a pole at $t = \infty$

Proposition 1.2. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x has a pole at $t = \infty$ and y, z, w are all holomorphic at $t = \infty$.*

Proof. Suppose that $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has such a solution. We then note that $n_0 \geq 1$, $n_1, n_2, n_3 \leq 0$ and $a_{\infty, n_0} \neq 0$. Comparing the coefficients of the term t^{2n_0} in

$$tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t,$$

we have $0 = 2a_{\infty, n_0}^2 b_{\infty, 0} - a_{\infty, n_0}^2$, which implies that $b_{\infty, 0} = 1/2$.

On the other hand, by comparing the coefficients of the term t^{n_0} in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we obtain $0 = -2a_{\infty, n_0} b_{\infty, 0}^2 + 2a_{\infty, n_0} b_{\infty, 0}$, which implies that $b_{\infty, 0} = 0, 1$. This is impossible. \square

1.2.2 The case where y has a pole at $t = \infty$

Lemma 1.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution near $t = \infty$ such that y has a pole of order n ($n \geq 1$) at $t = \infty$ and x is holomorphic at $t = \infty$. Then,*

$$a_{\infty, 0} = a_{\infty, -1} = \cdots = a_{\infty, -(n-1)} = 0, \quad a_{\infty, -n} = \frac{-n - \alpha_0 - \alpha_1}{2b_{\infty, n}}, \quad b_{\infty, n} \neq 0,$$

which implies that

$$2\alpha_3z + 2z^2w + t = tx' - \{2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x\} = O(t^{-1}).$$

Proof. Substituting the Laurent series of x, y at $t = \infty$ in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we can obtain the lemma. \square

By Lemma 1.3, we can easily prove the following proposition:

Proposition 1.4. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$.*

1.2.3 The case where z has a pole at $t = \infty$

Proposition 1.5. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$.*

(1) *If $n \geq 2$, then*

$$\begin{cases} x = (\alpha_0 - \alpha_1) - \frac{\{(n-1) + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}\{(n-1) + 2\alpha_3 + 2\alpha_4\}}{c_{\infty, n}} t^{-n} + \dots, \\ y = \frac{1}{2} + \frac{(n-1) + 2\alpha_3 + 2\alpha_4}{2c_{\infty, n}} t^{-n} + \dots, \\ z = c_{\infty, n} t^n + c_{\infty, n-1} t^{n-1} + \dots, \\ w = -\frac{\alpha_3}{c_{\infty, n}} t^{-n} + \dots. \end{cases}$$

(2) *If $n = 1$, then $\alpha_4 \neq 0$ and*

$$\begin{cases} x = (\alpha_0 - \alpha_1) - 2\alpha_4(2\alpha_2 + 2\alpha_3 + 2\alpha_4)(2\alpha_3 + 2\alpha_4)t^{-1} \dots, \\ y = \frac{1}{2} + 2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots, \\ z = \frac{1}{2\alpha_4}t + \dots, \\ w = -2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots. \end{cases}$$

Proof. It can be proved by direct calculation. □

From Proposition 1.5, let us consider the relationship between case (1) and the Bäcklund transformation, s_3 .

Corollary 1.6. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that $\alpha_3 \neq 0$. $s_3(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$.*

Proof. By direct calculation, we find that $s_3(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that $s_3(z)$ has a pole of order m ($1 \leq m \leq n-1$) at $t = \infty$ and all of $s_3(x, y, w)$ are holomorphic at $t = \infty$.

We assume that $s_3(z)$ has a pole of order m ($2 \leq m \leq n-1$) at $t = \infty$ and show a contradiction. By the definition of s_3 , we see that $s_3(w) = -\alpha_3/c_{\infty,n}t^{-n} + \dots$. On the other hand, we observe that

$$s_3(z) = c'_{\infty,m}t^m + \dots, \quad s_3(w) = -(-\alpha_3)/c'_{\infty,m}t^{-m} + \dots.$$

It then follows that

$$s_3(w) = -\alpha_3/c_{\infty,n}t^{-n} + \dots = -(-\alpha_3)/c'_{\infty,m}t^{-m} + \dots,$$

which is a contradiction. □

By Corollary 1.6, we can obtain the necessary conditions for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ to have a solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$.

Corollary 1.7. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. One of the following then occurs: (1) $\alpha_4 = 0$, (2) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1/2$.*

Proof. Let us first prove that $\alpha_4 = 0$ if $\alpha_3 \neq 0$. For this purpose, we note that $s_3(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, w)$ are holomorphic at $t = \infty$. By direct calculation and Proposition 1.5, we then see that $\alpha_4 + \alpha_3 \neq 0$ and

$$-\alpha_3/c_{\infty,n} \cdot t^{-n} + \dots = s_3(w) = -2(\alpha_4 + \alpha_3)\alpha_4 t^{-1} + \dots,$$

which implies that $\alpha_4 = 0$.

Let us show that $\alpha_4 = 0$ if $\alpha_3 = 0$ and $\alpha_2 \neq 0$. For this purpose, we note that $s_2(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_2, \alpha_4)$ such that $s_2(z)$ has a pole of order n at $t = \infty$ and all of $s_2(x, y, w)$ are holomorphic at $t = \infty$. Based on the above discussion, it then follows that $\alpha_4 = 0$.

If $\alpha_2 = \alpha_3 = 0$ and $\alpha_0 \neq 0$, or if $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 \neq 0$, we can show that $\alpha_4 = 0$ by using s_0 or s_1 .

The remaining case is that $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = 1/2$, which proves the corollary. □

Let us treat the case where z has a pole of order one at $t = \infty$.

Proposition 1.8. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order one at $t = \infty$ and x, y, w are holomorphic at $t = \infty$. It is then unique.*

Proof. Let us prove that the coefficients $a_{\infty,-2}, b_{\infty,-2}, c_{\infty,0}, d_{\infty,-2}$ are uniquely determined. The coefficients, $a_{\infty,-k}, b_{\infty,-k}, c_{\infty,-(k-2)}, d_{\infty,-k}$ ($k = 3, 4, \dots$) can be computed in the same way.

Comparing the constant terms in

$$tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t,$$

we have

$$d_{\infty,-2} = 1/\{2c_{\infty,1}^2\} \cdot \{(2\alpha_3 + 4\alpha_4)c_{\infty,0} - (\alpha_0 + \alpha_1)a_{\infty,0}\}, \quad (1.1)$$

where $a_{\infty,0}, c_{\infty,1}$ both have been determined. Comparing the coefficients of the term t^{-2} in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we obtain

$$a_{\infty,-2} = 2(\alpha_0 + \alpha_1 - 2)b_{\infty,-2} + 4a_{\infty,0}b_{\infty,-1}^2, \quad (1.2)$$

where $b_{\infty,-1}$ has been determined. Comparing the constant terms in

$$tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we have $b_{\infty,-2} + d_{\infty,-2} = 1/\{2c_{\infty,1}^2\} \cdot \{(2\alpha_4 - 1)c_{\infty,0}\}$, which implies that

$$b_{\infty,-2} = 1/\{2c_{\infty,1}^2\} \cdot \{(-2\alpha_3 - 2\alpha_4 - 1)c_{\infty,0} + (\alpha_0 + \alpha_1)a_{\infty,0}\}. \quad (1.3)$$

Moreover, comparing the coefficients of the term t^{-2} in

$$tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3,$$

we obtain

$$2c_{\infty,0}d_{\infty,-1}^2 + (1 + 2\alpha_4)d_{\infty,-2} + (2\alpha_3 + 4\alpha_4)b_{\infty,-2}. \quad (1.4)$$

From (1.1), (1.3) and (1.4), we have

$$c_{\infty,0} = (\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)(\alpha_0 + \alpha_1 + 2\alpha_2)/4\alpha_4^2,$$

which determines $a_{\infty,-2}, b_{\infty,-2}, d_{\infty,-2}$. □

1.2.4 The case where w has a pole at $t = \infty$

Proposition 1.9. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that w has a pole at $t = \infty$ and x, y, z are holomorphic at $t = \infty$.*

Proof. It can be proved in the same way as Proposition 1.4. □

1.3 The case where two of (x, y, z, w) have a pole at $t = \infty$

In this subsection, we consider the following four cases:

- (1) x, y have a pole at $t = \infty$ and z, w are holomorphic at $t = \infty$,
- (2) x, z have a pole at $t = \infty$ and y, w are holomorphic at $t = \infty$,
- (3) x, w have a pole at $t = \infty$ and y, z are holomorphic at $t = \infty$,
- (4) y, z have a pole at $t = \infty$ and x, w are holomorphic at $t = \infty$,
- (5) y, w have a pole at $t = \infty$ and x, z are holomorphic at $t = \infty$,
- (6) z, w have a pole at $t = \infty$ and x, y are holomorphic at $t = \infty$.

1.3.1 The case where x, y have a pole at $t = \infty$

Proposition 1.10. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, y have a pole at $t = \infty$ and z, w are holomorphic at $t = \infty$.*

Proof. Suppose that $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has such a solution. We then note that $n_0, n_1 \geq 1$, $n_1, n_2 \leq 0$ and $a_{\infty, n_0}, b_{\infty, n_1} \neq 0$.

On the other hand, comparing the coefficients of the term $t^{n_0+2n_1}$ in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we have $0 = -2a_{\infty, n_0}b_{\infty, n_1}^2$, which is impossible. \square

1.3.2 The case where x, z have a pole at $t = \infty$

Proposition 1.11. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, z have a pole at $t = \infty$ and y, w are holomorphic at $t = \infty$.*

Proof. Suppose that $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has such a solution. We then note that $n_0, n_2 \geq 1$, $n_1, n_3 \leq 0$, and $a_{\infty, n_0}, c_{\infty, n_2} \neq 0$.

Comparing the coefficients of the term t^{2n_2} and t^{n_2} in

$$\begin{cases} tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t, \\ tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3, \end{cases}$$

we have $b_{\infty, 0} + d_{\infty, 0} = 1/2$ and $-2d_{\infty, 0}^2 + 2d_{\infty, 0} - 4b_{\infty, 0}d_{\infty, 0} = 0$, which implies that $b_{\infty, 0} = 1/2, d_{\infty, 0} = 0$.

On the other hand, comparing the coefficients of the term t^{n_0} in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we have $0 = -2a_{\infty, n_0}b_{\infty, 0}^2 + 2a_{\infty, n_0}b_{\infty, 0}$, which implies that $b_{\infty, 0} = 0, 1$. This is impossible. \square

1.3.3 The case where x, w have a pole at $t = \infty$

Proposition 1.12. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, w have a pole at $t = \infty$ and y, z are holomorphic at $t = \infty$.*

Proof. It can be proved in the same way as Proposition 1.4. □

1.3.4 The case where y, z have a pole at $t = \infty$

Proposition 1.13. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that y, z have a pole at $t = \infty$ and x, w are holomorphic at $t = \infty$.*

Proof. Suppose that $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has such a solution. We then note that $n_1, n_2 \geq 1$, $n_0, n_2 \leq 0$, and $b_{\infty, n_1}, c_{\infty, n_2} \neq 0$.

Comparing the coefficients of the term $t^{n_1+2n_2}$ in

$$tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we have $0 = 2b_{\infty, n_1}c_{\infty, n_2}^2$, which is impossible. □

1.3.5 The case where y, w have a pole at $t = \infty$

By Lemma 1.3, we obtain the following lemma:

Lemma 1.14. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$. w then has a pole of order n_3 at $t = \infty$, where n_3 is an odd number and $n_3 \geq 3$.*

By Lemma 1.14, we can prove the following proposition:

Proposition 1.15. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$.*

Proof. Let us assume that w has a pole of order $n_3 = 3$ at $t = \infty$. If $n_3 = 5, 7, 9, \dots$, the proposition can be proved in the same way.

By Lemma 1.3, we find that $2z^2w + t = O(t^{-1})$ and $c_{\infty, 0} = 0$, $2c_{\infty, -1}^2d_{\infty, 3} + 1 = 0$, which implies that $c_{\infty, -1} \neq 0$. By considering that

$$tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we then observe that y has a pole of order $n_1 = 1, 2$ at $t = \infty$. Thus, comparing the coefficients of the term t^5 in

$$tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3,$$

we see that $-2c_{\infty, -1}d_{\infty, 3}^2 = 0$, which is impossible. □

1.3.6 The case where z, w have a pole at $t = \infty$

Proposition 1.16. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that z, w both have a pole at $t = \infty$ and x, y are both holomorphic at $t = \infty$.*

Proof. $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has such a solution. We then note that $n_2, n_3 \geq 1$, $n_0, n_1 \leq 0$, and $c_{\infty, n_0}, d_{\infty, n_3} \neq 0$.

Comparing the coefficients of the term $t^{2n_2+n_3}$ in

$$tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we have $0 = 2c_{\infty, n_2}^2 d_{\infty, n_3}$, which is impossible. \square

1.4 The case where three of (x, y, z, w) have a pole at $t = \infty$

In this subsection, we consider the following four cases:

- (1) x, y, z all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$,
- (2) x, y, w all have a pole at $t = \infty$ and z is holomorphic at $t = \infty$,
- (3) x, z, w all have a pole at $t = \infty$ and y is holomorphic at $t = \infty$,
- (4) y, z, w all have a pole at $t = \infty$ and x is holomorphic at $t = \infty$.

1.4.1 The case where x, y, z have a pole at $t = \infty$

Proposition 1.17. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, y, z all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$.*

Proof. It can be easily checked. \square

1.4.2 The case where x, y, w have a pole at $t = \infty$

Proposition 1.18. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, y, w all have a pole at $t = \infty$ and z is holomorphic at $t = \infty$.*

Proof. It can be easily checked. \square

1.4.3 The case where x, z, w have a pole at $t = \infty$

Proposition 1.19. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that x, z, w all have a pole at $t = \infty$ and y is holomorphic at $t = \infty$.*

Proof. It can be easily checked. \square

1.4.4 The case where y, z, w have a pole at $t = \infty$

Proposition 1.20. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that y, z, w all have a pole at $t = \infty$ and x is holomorphic at $t = \infty$.*

Proof. It can be easily checked. □

1.5 The case where all of (x, y, z, w) have a pole at $t = \infty$

Proposition 1.21. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that all of (x, y, z, w) have a pole at $t = \infty$.*

Proof. It can be easily checked. □

1.6 Summary

Let us summarize the results in this section.

Proposition 1.22. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution at $t = \infty$. z then has a pole of order n ($n \geq 1$) at $t = \infty$ and all of x, y, w are holomorphic at $t = \infty$.*

(1) If $n \geq 2$, then

$$\begin{cases} x = (\alpha_0 - \alpha_1) - \frac{\{(n-1) + 2\alpha_2 + 2\alpha_3 + 2\alpha_4\}\{(n-1) + 2\alpha_3 + 2\alpha_4\}}{c_{\infty, n}} t^{-n} + \dots, \\ y = \frac{1}{2} + \frac{(n-1) + 2\alpha_3 + 2\alpha_4}{2c_{\infty, n}} t^{-n} + \dots, \\ z = c_{\infty, n} t^n + c_{\infty, n-1} t^{n-1} + \dots, \\ w = -\frac{\alpha_3}{c_{\infty, n}} t^{-n} + \dots, \end{cases}$$

and either of the following occurs: (i) $\alpha_4 = 0$, (ii) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1/2$.

(2) If $n = 1$, then $\alpha_4 \neq 0$ and x, y, z, w are uniquely expanded as follows:

$$\begin{cases} x = (\alpha_0 - \alpha_1) - 2\alpha_4(2\alpha_2 + 2\alpha_3 + 2\alpha_4)(2\alpha_3 + 2\alpha_4)t^{-1} \dots, \\ y = \frac{1}{2} + 2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots, \\ z = \frac{1}{2\alpha_4}t + \dots, \\ w = -2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots. \end{cases}$$

By the uniqueness, we can determine the rational solutions of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ if $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$ and $\alpha_4 \neq 0$.

Corollary 1.23. *If $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$ and $\alpha_4 \neq 0$, then for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that*

$$x \equiv 0, \quad y \equiv 1/2, \quad z = 1/(2\alpha_4) \cdot t, \quad w \equiv 0,$$

and it is unique.

Proof. The corollary follows from the direct calculation and Proposition 1.22. □

2 Meromorphic solutions at $t = 0$

In this section, we treat the meromorphic solutions near $t = 0$. We then obtain the following proposition in the same way as Proposition 1.22.

Proposition 2.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution at $t = 0$. One of the following then occurs:*

- (1) x, y, z, w are all holomorphic at $t = 0$,
- (2) z has a pole of order n ($n \geq 1$) at $t = 0$ and x, y, w are all holomorphic at $t = 0$,
- (3) y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$.

2.1 The case where x, y, z, w are all holomorphic at $t = 0$

Proposition 2.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$. Then, $a_{0,0} = 0, \alpha_0 - \alpha_1$.*

Proof. Comparing the constant terms in

$$\begin{cases} tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t, \\ ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1, \\ tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t, \\ tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3, \end{cases}$$

we have

$$2a_{0,0}^2b_{0,0} - a_{0,0}^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)a_{0,0} + 2\alpha_3c_{0,0} + 2c_{0,0}^2d_{0,0} = 0, \quad (2.1)$$

$$-2a_{0,0}b_{0,0}^2 + 2a_{0,0}b_{0,0} - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)b_{0,0} + \alpha_1 = 0, \quad (2.2)$$

$$2c_{0,0}^2d_{0,0} - c_{0,0}^2 + (1 - 2\alpha_4)c_{0,0} + 2b_{0,0}c_{0,0}^2 = 0, \quad (2.3)$$

$$-2c_{0,0}d_{0,0}^2 + 2c_{0,0}d_{0,0} - (1 - 2\alpha_4)d_{0,0} - 2\alpha_3b_{0,0} - 4b_{0,0}c_{0,0}d_{0,0} + \alpha_3 = 0, \quad (2.4)$$

respectively. From (2.1) and (2.2), we see that

$$a_{0,0}^2 b_{0,0} + 2\alpha_3 b_{0,0} c_{0,0} + 2b_{0,0} c_{0,0}^2 d_{0,0} + \alpha_1 a_{0,0}. \quad (2.5)$$

From (2.3) and (2.4), we find that

$$c_{0,0}^2 d_{0,0} - 2\alpha_3 b_{0,0} c_{0,0} - 2b_{0,0} c_{0,0}^2 d_{0,0} + \alpha_3 c_{0,0} = 0. \quad (2.6)$$

From (2.5) and (2.6), we then have

$$a_{0,0}^2 b_{0,0} + \alpha_1 a_{0,0} + c_{0,0}^2 d_{0,0} + \alpha_3 c_{0,0} = 0. \quad (2.7)$$

By (2.1) and (2.7), we obtain

$$a_{0,0} \{a_{0,0} - (\alpha_0 - \alpha_1)\} = 0, \quad (2.8)$$

which implies that $a_{0,0} = 0, \alpha_0 - \alpha_1$.

□

2.1.1 The case where $a_{0,0} = 0$

Let us assume that $a_{0,0} = 0$

Proposition 2.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = 0$. One of the following then occurs:*

- (1) $\alpha_0 = \alpha_1 = 0$ and $a_{0,0} = c_{0,0} = 0$, $2\alpha_3 b_{0,0} + (2\alpha_2 + 2\alpha_3)d_{0,0} = \alpha_3$,
- (2) $\alpha_0 = \alpha_1 = 0$ and $a_{0,0} = 0$, $c_{0,0}d_{0,0} = -\alpha_3$, $2\alpha_3 b_{0,0} - 2\alpha_2 d_{0,0} = \alpha_3$,
- (3) $\alpha_0 = \alpha_1 \neq 0$, $\alpha_4 = 1/2$ and $a_{0,0} = 0$, $b_{0,0} = 1/2$, $c_{0,0} = 0$,
- (4) $\alpha_0 + \alpha_1 \neq 0$, $\alpha_3 = 0$, $\alpha_4 = 1/2$ and $a_{0,0} = 0$, $b_{0,0} = \alpha_1/(\alpha_0 + \alpha_1)$, $c_{0,0} = 0$,
- (5) $\alpha_0 + \alpha_1 \neq 0$, $\alpha_4 \neq 1/2$ and

$$a_{0,0} = 0, \quad b_{0,0} = \alpha_1/(\alpha_0 + \alpha_1), \quad c_{0,0} = 0, \quad d_{0,0} = -\alpha_3(\alpha_0 - \alpha_1)/\{(2\alpha_4 - 1)(\alpha_0 + \alpha_1)\},$$

- (6) $\alpha_0 = \alpha_1 \neq 0$, $\alpha_3 + \alpha_4 = 1/2$ and $a_{0,0} = 0$, $b_{0,0} = 1/2$, $c_{0,0}d_{0,0} = -\alpha_3$,
- (7) $\alpha_0 + \alpha_1 \neq 0$, $\alpha_0 - \alpha_1 \neq 0$, and

$$a_{0,0} = 0, \quad b_{0,0} = \frac{\alpha_1}{(\alpha_0 + \alpha_1)}, \quad c_{0,0} = \frac{(\alpha_0 + \alpha_1)(1 - 2\alpha_3 - 2\alpha_4)}{(\alpha_0 - \alpha_1)} \neq 0, \quad d_{0,0} = -\frac{\alpha_3(\alpha_0 - \alpha_1)}{(\alpha_0 + \alpha_1)(1 - 2\alpha_3 - 2\alpha_4)}.$$

Proof. It can be easily checked.

□

2.1.2 The case where $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$

Let us treat the case in which $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$. We first obtain the following proposition:

Proposition 2.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that all of (x, y, z, w) are holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$. Then, $b_{0,0} = 1/2$, $-\alpha_1/(\alpha_0 - \alpha_1)$.*

Proof. Substituting $a_{0,0} = \alpha_0 - \alpha_1$ in (2.2), we can obtain $b_{0,0} = 1/2$, $-\alpha_1/(\alpha_0 - \alpha_1)$. \square

Let us deal with the case where $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = 1/2$.

Proposition 2.5. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that all of (x, y, z, w) are holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$ and $b_{0,0} = 1/2$. One of the following then occurs:*

- (1) $\alpha_0 + \alpha_1 = 0$, $\alpha_4 = 1/2$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = 1/2$, $c_{0,0} = 0$,
- (2) $\alpha_0 + \alpha_1 = 0$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = 1/2$, $c_{0,0} = d_{0,0} = 0$,
- (3) $\alpha_0 + \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and

$$a_{0,0} = \alpha_0 - \alpha_1 \neq 0, b_{0,0} = 1/2, c_{0,0} \neq 0, d_{0,0} = (2\alpha_4 - 1)/\{2c_{0,0}\},$$

- (4) $\alpha_3 + \alpha_4 \neq 1/2$ and

$$a_{0,0} = \alpha_0 - \alpha_1, b_{0,0} = \frac{1}{2}, c_{0,0} = \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{1 - 2\alpha_3 - 2\alpha_4} \neq 0, d_{0,0} = -\frac{(1 - 2\alpha_4)(1 - 2\alpha_3 - 2\alpha_4)}{2(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}.$$

Proof. It can be easily checked. \square

Let us treat the case where $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} \neq 1/2$.

Proposition 2.6. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that all of (x, y, z, w) are holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$ and $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1) \neq 1/2$, which implies that $\alpha_0 + \alpha_1 \neq 0$. One of the following then occurs:*

- (1) $\alpha_3 = 0$, $\alpha_4 = 1/2$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1)$, $c_{0,0} = 0$,
- (2) $\alpha_4 \neq 1/2$ and

$$a_{0,0} = \alpha_0 - \alpha_1, b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1), c_{0,0} = 0, d_{0,0} = \alpha_3(\alpha_0 + \alpha_1)/\{(1 - 2\alpha_4)(\alpha_0 - \alpha_1)\},$$

- (3) $\alpha_3 = 0$ and $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1)$, $c_{0,0} \neq 0$, $d_{0,0} = 0$,
- (4) $\alpha_3 \neq 0$ and

$$a_{0,0} = \alpha_0 - \alpha_1, b_{0,0} = \frac{-\alpha_1}{\alpha_0 - \alpha_1}, c_{0,0} = \frac{(\alpha_0 - \alpha_1)(1 - 2\alpha_3 - 2\alpha_4)}{\alpha_0 + \alpha_1} \neq 0, d_{0,0} = \frac{-\alpha_3(\alpha_0 + \alpha_1)}{(\alpha_0 - \alpha_1)(1 - 2\alpha_3 - 2\alpha_4)}.$$

Proof. It can be easily checked. □

2.2 The case where z has a pole at $t = 0$

Proposition 2.7. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = 0$ and x, y, w are all holomorphic at $t = 0$. Then,*

$$\begin{cases} x = (\alpha_0 - \alpha_1) - \frac{\{(n+1) - 2\alpha_3 - 2\alpha_4\}\{(n+1) - 2\alpha_2 - 2\alpha_3 - 2\alpha_4\}}{c_{0,-n}} t^n + \dots, \\ y = \frac{1}{2} - \frac{(n+1) - 2\alpha_3 - 2\alpha_4}{2c_{0,-n}} t^n + \dots, \\ z = c_{0,-n} t^{-n} + c_{0,-(n-1)} t^{-(n-1)} + \dots, \\ w = -\frac{\alpha_3}{c_{0,-n}} t^n + \dots. \end{cases}$$

Proof. It can be easily checked. □

Let us study the relationship between the solution in Proposition 2.7 and the Bäcklund transformation, s_3 .

Corollary 2.8. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = 0$ and x, y, w are all holomorphic at $t = 0$. Moreover, assume that $\alpha_3 \neq 0$. $s_3(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$.*

Proof. By direct calculation, we see that $s_3(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that $s_3(z)$ has a pole of order m ($0 \leq m \leq n - 1$) at $t = 0$ and all of $s_3(x, y, w)$ are holomorphic at $t = 0$.

We assume that $s_3(z)$ has a pole of order m ($1 \leq m \leq n - 1$) at $t = 0$, and show contradiction. From Proposition 2.7, it follows that

$$s_3(z) = c'_{0,-m} t^{-m} + \dots, \quad s_3(w) = -(-\alpha_3)/c'_{0,-m} t^m + \dots.$$

On the other hand, by the definition of s_3 , we find that $s_3(w) = -\alpha_3/c_{0,-n} t^n + \dots$, which is a contradiction. □

2.3 The case where y, w have a pole at $t = 0$

Let us prove the following four lemmas:

Lemma 2.9. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y has a pole of order n ($n \geq 1$) at $t = 0$ and x is holomorphic at $t = 0$. Then,*

$$a_{0,0} = a_{0,1} = \cdots = a_{0,(n-1)} = 0, \quad a_{0,n} = \frac{(n-1) + 2\alpha_2 + 2\alpha_3 + 2\alpha_4}{2b_{0,-n}},$$

which implies that

$$2\alpha_3 z + 2z^2 w + t = tx' - [2x^2 y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x] = O(t^2).$$

Proof. Substituting the Laurent series of x, y at $t = 0$ in

$$ty' = -2xy^2 + 2xy - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y + \alpha_1,$$

we can prove the proposition. \square

Lemma 2.10. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole of order n_1, n_3 ($n_1, n_3 \geq 1$) at $t = 0$, and x, z are both holomorphic at $t = 0$. n_3 is then an odd number.*

Proof. We suppose that $n_3 = 2$ and show a contradiction. If $n_3 = 4, 6, \dots$, we can prove the contradiction in the same way.

Comparing the coefficients of the terms t^{-2}, t^0 in

$$tx' = 2x^2 y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3 z + 2z^2 w + t,$$

we find that $c_{0,0} = c_{0,1} = 0$. Furthermore, by comparing the coefficients of the term t in

$$tx' = 2x^2 y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3 z + 2z^2 w + t,$$

we see that $0 = 1$, which is impossible. \square

Lemma 2.11. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole of order n_1, n_3 ($n_1, n_3 \geq 1$) at $t = 0$, and x, z are both holomorphic at $t = 0$. Moreover, assume that $n_3 = 3, 5, 7, \dots$. Then, $n_1 \leq n_3$.*

Proof. We treat the case where $n_3 = 3$. The other cases can be proved in the same way.

Comparing the coefficients of the terms t^{-3}, t^{-1} and t in

$$tx' = 2x^2 y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3 z + 2z^2 w + t,$$

we observe that $c_{0,0} = c_{0,1} = 0$ and $2c_{0,2}^2 d_{0,-3} + 1 = 0$, which implies that $c_{0,2} \neq 0$.

Comparing the lowest terms in

$$tz' = 2z^2 w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we find that $n_1 = 1, 2, 3$. \square

Lemma 2.12. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$. y, w then both have a pole of order one at $t = 0$.*

Proof. We assume that $n_3 = 3$ and show a contradiction. If $n_3 = 5, 7, \dots$, we can prove the contradiction in the same way.

By Lemma 2.11 and its proof, we see that $c_{0,0} = c_{0,1} = 0$, $2c_{0,2}^2 d_{0,-3} + 1 = 0$ and $n_1 = 1, 2, 3$. We note that $c_{0,2} \neq 0$.

Let us first suppose that $n_1 = 1$ and $n_3 = 3$. By comparing the coefficients of the terms t in

$$tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t,$$

we then find that $2c_{0,2}^2 d_{0,-3} + 2b_{0,-3}c_{0,2}^2 + 1 = 0$, which implies that $2b_{0,-3}c_{0,2}^2 = 0$. This is impossible.

Let us suppose that $n_1 = 1, 2$ and n_3 . By comparing the coefficients of the terms t^{-4} in

$$tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3,$$

we then observe that $-2c_{0,2}d_{0,-3}^2 = 0$, which is impossible. \square

Let us then prove the following proposition:

Proposition 2.13. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$. Then, $\alpha_4(\alpha_3 + \alpha_4) \neq 0$ and*

$$\begin{cases} x = \frac{\alpha_2 + \alpha_3 + \alpha_4}{2\alpha_4(\alpha_3 + \alpha_4)}t + \dots, \\ y = 2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots, \\ z = \frac{1}{2\alpha_4}t + \dots, \\ w = -2\alpha_4(\alpha_3 + \alpha_4)t^{-1} + \dots. \end{cases}$$

Proof. Let us first note that y, w have a pole of order one at $t = 0$. From Lemma 2.9, it then follows that $c_{0,0} = 0$.

Comparing the terms t, t, t^{-1} in

$$\begin{cases} tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t, \\ tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t, \\ tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w - 2\alpha_3y - 4yzw + \alpha_3, \end{cases}$$

we find that

$$\begin{cases} 2\alpha_3 c_{0,1} + 2c_{0,1}^2 d_{0,-1} + 1 = 0, \\ 2c_{0,1}^2 - 2\alpha_4 c_{0,1} + 2b_{0,-1} c_{0,1}^2 + 1 = 0, \\ -2c_{0,1} d_{0,-1}^2 + 2\alpha_4 d_{0,-1} - 2\alpha_3 b_{0,-1} - 4b_{0,-1} c_{0,1} d_{0,-1} = 0, \end{cases} \quad (2.9)$$

respectively. Based on the first equation of (2.9), we find that $c_{0,1} \neq 0$. The first and second equations of (2.9) shows that $2(\alpha_3 + \alpha_4)c_{0,1} - 2b_{0,-1}c_{0,1}^2 = 0$, which implies that

$$c_{0,1} = (\alpha_3 + \alpha_4)/b_{0,-1} \neq 0.$$

Thus, from the first equation of (2.9), we find that

$$d_{0,-1} = -b_{0,-1}^2 / \{2(\alpha_3 + \alpha_4)^2\} - \alpha_3 b_{0,-1} / (\alpha_3 + \alpha_4).$$

From the third equation of (2.9), we then see that $b_{0,-1} = 2\alpha_4(\alpha_3 + \alpha_4) \neq 0$.

By Lemma 2.9, we observe that

$$\begin{cases} a_{0,0} = 0, & a_{0,1} = (\alpha_2 + \alpha_3 + \alpha_4) / \{2\alpha_4(\alpha_3 + \alpha_4)\}, \\ b_{0,-1} = 2\alpha_4(\alpha_3 + \alpha_4), \\ c_{0,0} = 0, & c_{0,1} = 1/2 \cdot \alpha_4, \\ d_{0,-1} = -2\alpha_4(\alpha_3 + \alpha_4). \end{cases}$$

□

2.4 Summary

Let us summarize the results in this section.

Proposition 2.14. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution at $t = 0$. Then, $a_{0,0} = 0$, $\alpha_0 - \alpha_1$ and $y + w$ is holomorphic at $t = 0$.*

3 Meromorphic solution at $t = c \in \mathbb{C}^*$

In this section, we treat meromorphic solutions near $t = c \in \mathbb{C}^*$. We then obtain the following proposition in the same way as Proposition 1.22.

Proposition 3.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution such that some of (x, y, z, w) have a pole at $t = c \in \mathbb{C}^*$. One of the following then occurs:*

- (1) *x has a pole of order one at $t = c$ and y, z, w are all holomorphic at $t = c$,*

- (2) y has a pole of order two at $t = c$ and x, z, w are all holomorphic at $t = c$,
- (3) z has a pole of order n ($n \geq 1$) at $t = c$ and x, y, w are all holomorphic at $t = c$,
- (4) w has a pole of order two at $t = c$ and y, z, w are all holomorphic at $t = c$,
- (5) x, z both have a pole of order one at $t = c$ and y, w are both holomorphic at $t = c$,
- (6) x has a pole of order one at $t = c$ and w has a pole of order two at $t = c$ and y, z both are holomorphic at $t = c$,
- (7) y, w both have a pole at $t = c$ and x, z are both holomorphic at $t = c$.

3.1 The case where x has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x has a pole of order one at $t = c \in \mathbb{C}^*$ and y, z, w are all holomorphic at $t = c$. Then, $b_{c,0} = 0, 1$.*

- (1) If $b_{c,0} = 0$,

$$\begin{cases} x = c(t - c)^{-1} + \cdots, \\ y = -\frac{\alpha_1}{c}(t - c) + \cdots. \end{cases}$$

- (2) If $b_{c,0} = 1$,

$$\begin{cases} x = -c(t - c)^{-1} + \cdots, \\ y = 1 + \frac{\alpha_0}{c}(t - c) + \cdots. \end{cases}$$

Proof. It can be easily checked. □

3.2 The case where y has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y has a pole of order two at $t = c \in \mathbb{C}^*$ and x, z, w are all holomorphic at $t = c$. Then, $\alpha_3 = 0$ and one of the following occurs:*

$$(1) \begin{cases} x = -(t - c) + \frac{(\alpha_0 + \alpha_1) - 2}{2c}(t - c)^2 + \cdots, \\ y = -c(t - c)^{-2} + b_{c,0} + \cdots, \\ z = \frac{1}{2}(t - c) + \cdots, \\ w = d_{c,2}(t - c)^2 + \cdots, \end{cases}$$

$$(2) \begin{cases} x = -(t - c) + \frac{(\alpha_0 + \alpha_1) - 2}{2c}(t - c)^2 \dots, \\ y = -c(t - c)^{-2} + b_{c,0} + \dots, \\ z = -(t - c) + \dots, \\ w = d_{c,2}(t - c)^2 + \dots. \end{cases}$$

Proof. It can be easily checked. □

3.3 The case where z has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = c \in \mathbb{C}^*$ and x, y, w are all holomorphic at $t = c$.*

(1) If $n \geq 2$,

$$\begin{cases} y = \frac{1}{2} + O((t - c)^{n-1}) \dots, \\ z = c_{0,-n}(t - c)^{-n} + \dots, \\ w = -\frac{\alpha_3}{c_{0,-n}}(t - c)^n + \dots. \end{cases}$$

(2) If $n = 1$, then

$$\begin{cases} y = \left(\frac{1}{2} - \frac{c}{2c_{c,-1}} \right) + \dots, \\ z = c_{c,-1}(t - c)^{-1} + \dots, \\ w = -\frac{\alpha_3}{c_{c,-1}}(t - c) + \dots. \end{cases}$$

Proof. It can be easily checked. □

3.4 The case where w has a pole at $t = c \in \mathbb{C}^*$

Proposition 3.5. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that w has a pole of order two at $t = c \in \mathbb{C}^*$ and x, y, z are all holomorphic at $t = c$. Then,*

$$\begin{cases} y = b_{c,0} + \dots, \\ z = -(t - c) - \frac{2\alpha_4 + 1}{2c}(t - c)^2 + c_{c,3}(t - c)^3 + \dots, \\ w = -c(t - c)^{-2} + d_{c,0} + \dots, \end{cases}$$

where $b_{c,0} + d_{c,0} = (1 - c_{c,3}c)/2$.

Proof. It can be easily checked. □

3.5 The case where x, z have a pole at $t = c \in \mathbb{C}^*$

Proposition 3.6. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, z have a pole of order one at $t = c \in \mathbb{C}^*$ and y, w are all holomorphic at $t = c$. Then, $(b_{c,0}, d_{c,0}) = (0, 0), (0, 1), (1, 0), (1, -1)$.*

(1) If $(b_{c,0}, d_{c,0}) = (0, 0)$,

$$\begin{cases} x = c(t-c)^{-1} + \dots, \\ y = -\frac{\alpha_1}{c}(t-c) + \dots, \\ z = c(t-c)^{-1} + \dots, \\ w = -\frac{\alpha_3}{c}(t-c) + \dots. \end{cases}$$

(2) Assume that $(b_{c,0}, d_{c,0}) = (0, 1)$. Then, $(a_{c,-1}, c_{c,-1}) = (-c, -c), (2c, -c)$.

(i) If $(a_{c,-1}, c_{c,-1}) = (-c, -c)$,

$$\begin{cases} x = -c(t-c)^{-1} + \dots, \\ y = \frac{\alpha_1}{3c}(t-c) + \dots, \\ z = -c(t-c)^{-1} + \dots, \\ w = 1 + \left(1 - \frac{4\alpha_1}{3} - \alpha_3 - 2\alpha_4\right) \frac{1}{c}(t-c) + \dots. \end{cases}$$

(ii) If $(a_{c,-1}, c_{c,-1}) = (2c, -c)$,

$$\begin{cases} x = 2c(t-c)^{-1} + \dots, \\ y = -\frac{\alpha_1}{3c}(t-c) + \dots, \\ z = -c(t-c)^{-1} + \dots, \\ w = 1 + \left(1 + \frac{4\alpha_1}{3} - \alpha_3 - 2\alpha_4\right) \frac{1}{c}(t-c) + \dots. \end{cases}$$

(3) If $(b_{c,0}, d_{c,0}) = (1, 0)$,

$$\begin{cases} x = -c(t-c)^{-1} + \dots, \\ y = 1 + \frac{\alpha_0}{c}(t-c) + \dots, \\ z = -c(t-c)^{-1} + \dots, \\ w = \frac{\alpha_3}{c}(t-c) + \dots. \end{cases}$$

(4) Assume that $(b_{c,0}, d_{c,0}) = (1, -1)$. Then, $(a_{c,-1}, c_{c,-1}) = (c, c), (-2c, c)$.

(i) If $(a_{c,-1}, c_{c,-1}) = (c, c)$,

$$\begin{cases} x = c(t-c)^{-1} + \dots, \\ y = 1 - \frac{\alpha_0}{3c}(t-c) + \dots, \\ z = c(t-c)^{-1} + \dots, \\ w = -1 + \left(-1 + \frac{4\alpha_0}{3} + \alpha_3 + 2\alpha_4\right) \frac{1}{c}(t-c) + \dots. \end{cases}$$

(ii) If $(a_{c,-1}, c_{c,-1}) = (-2c, c)$,

$$\begin{cases} x = -2c(t-c)^{-1} + \dots, \\ y = 1 + \frac{\alpha_0}{3c}(t-c) + \dots, \\ z = c(t-c)^{-1} + \dots, \\ w = -1 + \left(-1 - \frac{4\alpha_0}{3} + \alpha_3 + 2\alpha_4\right) \frac{1}{c}(t-c) + \dots. \end{cases}$$

Proof. It can be easily checked. □

3.6 The case where x, w have a pole at $t = c \in \mathbb{C}^*$

Proposition 3.7. Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, w have a pole at $t = c \in \mathbb{C}^*$ and y, z are all holomorphic at $t = c$. Then, $b_{c,0} = 0, 1$.

(1) If $b_{c,0} = 0$,

$$\begin{cases} x = c(t-c)^{-1} + \frac{1 + \alpha_0 - \alpha_1}{2} + \dots, \\ y = -\frac{\alpha_1}{c}(t-c) + \dots, \\ z = -(t-c) - \frac{2\alpha_4 + 1}{2c}(t-c)^2 + c_{c,3}(t-c)^3 + \dots, \\ w = -c(t-c)^{-2} + \frac{1 - c_{c,3}c}{2} + \dots. \end{cases}$$

(2) If $b_{c,0} = 1$,

$$\begin{cases} x = -c(t-c)^{-1} + \frac{-1 + \alpha_0 - \alpha_1}{2} + \dots, \\ y = 1 + \frac{\alpha_0}{c}(t-c) + \dots, \\ z = -(t-c) - \frac{2\alpha_4 + 1}{2c}(t-c)^2 + c_{c,3}(t-c)^3 + \dots, \\ w = -c(t-c)^{-2} + \frac{-1 - c_{c,3}c}{2} + \dots. \end{cases}$$

Proof. It can be easily checked. □

3.7 The case where y, w have a pole at $t = c \in \mathbb{C}^*$

In the same way as Proposition 2.13, we can prove the following proposition:

Proposition 3.8. *Suppose that for $B_4(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y has a pole of order two at $t = c \in \mathbb{C}^*$ and w has a pole of order one at $t = c$ and x, z are holomorphic at $t = c$. Then, $c_{c,0} = 0$ and $c_{c,1} = 1/2, -1$.*

(1) If $c_{c,1} = 1/2$,

$$\begin{cases} x = -(t - c) + \frac{\alpha_0 + \alpha_1 - 2\alpha_3 - 2}{2c}(t - c)^2 + \dots, \\ y = -c(t - c)^{-2} + \frac{2\alpha_3}{3}(t - c)^{-1} + \dots, \\ z = \frac{1}{2}(t - c) + \frac{-\alpha_4 + 1}{4c}(t - c)^2 + \dots, \\ w = -\frac{2\alpha_3}{3}(t - c)^{-1} + \frac{\alpha_3(4\alpha_3 + 3\alpha_4 + 3)}{9c} + \dots. \end{cases}$$

(2) If $c_{c,1} = -1$,

$$\begin{cases} x = -(t - c) + \frac{\alpha_0 + \alpha_1 + 2\alpha_3 - 2}{2c}(t - c)^2 + \dots, \\ y = -c(t - c)^{-2} - \frac{2\alpha_3}{3}(t - c)^{-1} + \dots, \\ z = -(t - c) + \frac{-2\alpha_4 - 1}{2c}(t - c)^2 + \dots, \\ w = \frac{2\alpha_3}{3}(t - c)^{-1} + \frac{\alpha_3(\alpha_3 - 3\alpha_4 - 3)}{9c} + \dots. \end{cases}$$

Proposition 3.9. *Suppose that for $B_4(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole of order two at $t = c \in \mathbb{C}^*$ and x, z are holomorphic at $t = c$. Then, $c_{c,0} = 0$ and $c_{c,1} = -1/2, 1$.*

(1) If $c_{c,1} = -1/2$,

$$\begin{cases} x = (t - c) + \frac{1 + 2\alpha_2 + 4\alpha_3 + 6\alpha_4}{2c}(t - c)^2 + \dots, \\ y = c(t - c)^{-2} - \frac{2\alpha_3 + 4\alpha_4}{3}(t - c)^{-1} + \dots, \\ z = -\frac{1}{2}(t - c) - \frac{\alpha_4 + 1}{4c}(t - c)^2 + \dots, \\ w = -4c(t - c)^{-2} + \frac{2\alpha_3 + 4\alpha_4}{3}(t - c)^{-1} + \dots. \end{cases}$$

(2) If $c_{c,1} = 1$,

$$\begin{cases} x = (t - c) + \frac{1 + 2\alpha_2 - 2\alpha_4}{2c}(t - c)^2 + \dots, \\ y = c(t - c)^{-2} + \frac{2\alpha_3 + 4\alpha_4}{3}(t - c)^{-1} + \dots, \\ z = (t - c) + \frac{1 - 2\alpha_4}{2c}(t - c)^2 + \dots, \\ w = -c(t - c)^{-2} - \frac{2\alpha_3 + 4\alpha_4}{3}(t - c)^{-1} + \dots. \end{cases}$$

Proposition 3.10. Suppose that for $B_4(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w both have a pole of order one at $t = c \in \mathbb{C}^*$ and x, z are holomorphic at $t = c \in \mathbb{C}^*$. Then, $c_{c,0} \neq 0$ and

$$\begin{cases} x = \frac{c}{2b_{c,-1}} + a_{c,1}(t - c) + \dots, \\ y = b_{c,-1}(t - c)^{-1} + \dots, \\ z = \frac{c}{2b_{c,-1}} + c_{c,1}(t - c) + \dots, \\ w = d_{c,-1}(t - c)^{-1} + \dots, \end{cases}$$

where $b_{c,-1} + d_{c,-1} = 0$ and $a_{c,1} - c_{c,1} = 2\alpha_2 c_{c,0}$.

3.8 Summary

Proposition 3.11. (1) Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a meromorphic solution at $t = c \in \mathbb{C}^*$. (i) and (ii) then hold:

- (i) x has a pole of order at most one at $t = c$ and $\text{Res}_{t=c} x = nc, n \in \mathbb{Z}$,
- (ii) $y + w$ has a pole of order at most two at $t = c$ and $\text{Res}_{t=c}(y + w) = 0$.

(2) Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. Then,

$$a_{\infty,0} - a_{0,0} \in \mathbb{Z} \text{ and } b_{\infty,-1} + d_{\infty,-1} = 0,$$

because $b_{0,-1} + d_{0,-1} = b_{c,-1} + d_{c,-1} = 0$.

Proof. Case (1) is obvious. Case (2) can be proved by applying the residue theorem to $t^{-1}x$. □

4 Hamiltonian and its properties

4.1 The Laurent series of $H_{B_4^{(1)}}$ at $t = \infty$

By Proposition 1.22, we can calculate the constant term, $h_{\infty,0}$, of the Laurent series of $H_{B_4^{(1)}}$ at $t = \infty$.

Proposition 4.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole order n ($n \geq 1$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$.*

(1) *If $n \geq 2$, then*

$$h_{\infty,0} = \frac{1}{4}(\alpha_0 - \alpha_1)^2 + \alpha_3(\alpha_3 + 2\alpha_4 - 1).$$

(2) *If $n = 1$, then*

$$h_{\infty,0} = \frac{1}{4}(\alpha_0 - \alpha_1)^2 + (\alpha_3 + \alpha_4)^2 - (\alpha_3 + \alpha_4).$$

4.2 The Laurent series of $H_{B_4^{(1)}}$ at $t = 0$

In this subsection, we compute the constant term, $h_{0,0}$, of the Laurent series of $H_{B_4^{(1)}}$ at $t = 0$.

4.2.1 The case where all of (x, y, z, w) are holomorphic at $t = 0$

By Propositions 2.3, 2.5 and 2.6, we obtain the following proposition:

Proposition 4.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that all of (x, y, z, w) are holomorphic at $t = 0$. Then, $a_{0,0} = 0, \alpha_0 - \alpha_1$.*

(1)-(i) *If $a_{0,0} = 0$ and $c_{0,0} = 0$,*

$$h_{0,0} = 0.$$

(1)-(ii) *If $a_{0,0} = 0$ and $c_{0,0} \neq 0$,*

$$h_{0,0} = \alpha_3(\alpha_3 + 2\alpha_4 - 1).$$

(2) *Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$. Then, $b_{0,0} = 1/2, -\alpha_1/(\alpha_0 - \alpha_1)$.*

(2)-(i) *If $b_{0,0} = 1/2$ and $c_{0,0} = 0$,*

$$h_{0,0} = \frac{1}{4}(\alpha_0 - \alpha_1)^2.$$

(2)-(ii) *If $b_{0,0} = 1/2$ and $c_{0,0} \neq 0$,*

$$h_{0,0} = \frac{1}{4}(\alpha_0 - \alpha_1)^2 - \frac{1}{4}(2\alpha_4 - 1)^2.$$

(2)-(iii) If $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1) \neq 1/2$ and $c_{0,0} = 0$,

$$h_{0,0} = -\alpha_0\alpha_1.$$

(2)-(iv) If $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1) \neq 1/2$ and $c_{0,0} \neq 0$,

$$h_{0,0} = -\alpha_0\alpha_1 + \alpha_3(\alpha_3 + 2\alpha_4 - 1).$$

4.2.2 The case where z has a pole at $t = 0$

By Proposition 2.7, we obtain the following proposition:

Proposition 4.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = 0$ and x, y, w are all holomorphic at $t = 0$. Then,*

$$h_{0,0} = \frac{1}{4}(\alpha_0 - \alpha_1)^2 + \alpha_3(\alpha_3 + 2\alpha_4 - 1).$$

4.2.3 The case where y, w have a pole at $t = 0$

By Proposition 2.13, we have the following proposition:

Proposition 4.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y, w have a pole at $t = 0$ and x, z are holomorphic at $t = 0$. Then,*

$$h_{0,0} = \alpha_2(\alpha_0 + \alpha_1 + \alpha_2).$$

4.3 The Laurent series of $H_{B_4^{(1)}}$ at $t = c \in \mathbb{C}^*$

In this subsection, we calculate the residue, $h_{c,-1}$, of $H_{B_4^{(1)}}$ at $t = c \in \mathbb{C}^*$.

4.3.1 The case where x has a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.2, we can show the following proposition:

Proposition 4.5. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x has a pole at $t = c \in \mathbb{C}^*$ and y, z, w are all holomorphic at $t = c$. $H_{B_4^{(1)}}$ is then holomorphic at $t = c$.*

4.3.2 The case where y has a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.3, we can prove the following proposition:

Proposition 4.6. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that y has a pole of order two at $t = c \in \mathbb{C}^*$ and x, z, w are all holomorphic at $t = c$. Then, $\alpha_3 = 0$ and $H_{B_4^{(1)}}$ has a pole of order one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = c$.*

4.3.3 The case where z has a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.4, we can show the following proposition:

Proposition 4.7. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that z has a pole of order n ($n \geq 1$) at $t = c \in \mathbb{C}^*$ and x, y, w are all holomorphic at $t = c$. $H_{B_4^{(1)}}$ is then holomorphic at $t = c$.*

4.3.4 The case where w has a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.5, we can prove the following proposition:

Proposition 4.8. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that w has a pole of order two at $t = c \in \mathbb{C}^*$ and x, y, z are all holomorphic at $t = c$. $H_{B_4^{(1)}}$ then has a pole of order one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = c$.*

4.3.5 The case where x, z have a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.6, we have the following proposition:

Proposition 4.9. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that x, z have a pole of order one at $t = c \in \mathbb{C}^*$ and y, w are holomorphic at $t = c$. Then, $(b_{c,0}, d_{c,0}) = (0, 0), (0, 1), (1, 0), (1, -1)$.*

- (1) *If $(b_{c,0}, d_{c,0}) = (0, 0)$, $H_{B_4^{(1)}}$ is holomorphic at $t = c$.*
- (2) *If $(b_{c,0}, d_{c,0}) = (0, 1)$, $H_{B_4^{(1)}}$ is holomorphic at $t = c$.*
- (3) *If $(b_{c,0}, d_{c,0}) = (1, 0)$, $H_{B_4^{(1)}}$ is holomorphic at $t = c$.*
- (4) *If $(b_{c,0}, d_{c,0}) = (1, -1)$, $H_{B_4^{(1)}}$ is holomorphic at $t = c$.*

4.3.6 The case where x, w have a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.7, we have the following proposition:

Proposition 4.10. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that x, w have a pole at $t = c \in \mathbb{C}^*$ and y, z are holomorphic at $t = c$. $H_{B_4^{(1)}}$ then has a pole of order one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = c$.*

4.3.7 The case where y, w have a pole at $t = c \in \mathbb{C}^*$

By Proposition 3.8, we obtain the following proposition:

Proposition 4.11. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y has a pole order two at $t = c \in \mathbb{C}^*$ and w has a pole of order one at $t = c$ and x, z are holomorphic at $t = c$. $H_{B_4^{(1)}}$ then has a pole of order one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = c$.*

By Proposition 3.9, we have the following proposition:

Proposition 4.12. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y, w have a pole order two at $t = c \in \mathbb{C}^*$ and x, z are holomorphic at $t = c$. Then, $c_{c,1} = -1/2, 1$.*
(1) *If $c_{c,1} = -1/2$, $H_{B_4^{(1)}}$ has a pole of order one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = 3c$.*
(2) *If $c_{c,1} = 1$, $H_{B_4^{(1)}}$ is holomorphic at $t = c$.*

By Proposition 3.10, we have the following proposition:

Proposition 4.13. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a solution such that y, w both have a pole order one at $t = c \in \mathbb{C}^*$ and x, z are holomorphic at $t = c$. $H_{B_4^{(1)}}$ is then holomorphic at $t = c$.*

4.4 Summary

Proposition 4.14. (1) *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a meromorphic solution at $t = c \in \mathbb{C}^*$. $H_{B_4^{(1)}}$ then has a pole of order at most one at $t = c$ and $h_{c,-1} = \text{Res}_{t=c} H_{B_4^{(1)}} = nc$ ($n = 0, 1, 3$).*
(2) *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 5}$, there exists a rational solution. Then, $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$.*

Proof. Case (1) is obvious. Case (2) can be proved by applying the residue theorem to $t^{-1}H_{B_4^{(1)}}$

□

5 Bäcklund transformations and their properties

In this section, following Sasano [26], we introduce the Bäcklund transformations $s_0, s_1, s_2, s_3, s_4, \pi_1$, and π_2 and investigate their properties.

5.1 Definition of the Bäcklund transformations

$B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has Bäcklund transformations, $s_0, s_1, s_2, s_3, s_4, \pi_1$, and π_2 , which are defined by

$$\begin{aligned}
s_0 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x + \frac{\alpha_0}{y-1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4 \right), \\
s_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4 \right), \\
s_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \\
&\quad \left(x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 \right), \\
s_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3 \right), \\
s_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x, y, z, w - \frac{2\alpha_4}{z} + \frac{t}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4 \right), \\
\pi_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (-x, 1-y, -z, -w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4), \\
\pi_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \\
&\quad \left(\frac{t}{z}, -\frac{z}{t}(zw + \alpha_3), \frac{t}{x}, -\frac{x}{t}(xy + \alpha_1), t; 2\alpha_4 + \alpha_3, \alpha_3, \alpha_2, \alpha_1, \frac{\alpha_0 - \alpha_1}{2} \right).
\end{aligned}$$

Let us note that the Bäcklund transformations, except for π_1 , are not polynomial in x, y, z, w and t but rational in x, y, z, w and t .

Remark At first, Sasano [26] defined

$$\pi_2 : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longrightarrow (2\alpha_4 + \alpha_3, \alpha_3, \alpha_2, (\alpha_0 - \alpha_1)/2, \alpha_1).$$

But, following Sasano [32], we corrected and redefined it

$$\pi_2 : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \longrightarrow (2\alpha_4 + \alpha_3, \alpha_3, \alpha_2, \alpha_1, (\alpha_0 - \alpha_1)/2).$$

5.2 The properties of the Bäcklund transformations

Considering s_0, s_1, s_2, s_3 , we can prove the following proposition:

Proposition 5.1. (0) If $y \equiv 1$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_0 = 0$.

- (1) If $y \equiv 0$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_1 = 0$.
- (2) If $x \equiv z$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_2 = 0$.
- (3) If $w \equiv 0$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_3 = 0$ or $y \equiv 1/2$.
- (4) For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that $z \equiv 0$.

Proof. We treat case (2). The other cases can be proved in the same way. By considering

$$\begin{cases} tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t, \\ tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + 2yz^2 + t, \end{cases}$$

we find that $\alpha_2 = 0$ or $x \equiv z \equiv 0$. If $x \equiv z \equiv 0$, it follows that $0 = t$, because

$$tx' = 2x^2y - x^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x + 2\alpha_3z + 2z^2w + t,$$

but this is impossible. □

By Proposition 5.1, we consider s_0 , s_1 , or s_2 as the identical transformation, if $y \equiv 1$, $y \equiv 0$ or $x \equiv z$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$.

Considering s_3 more in detail, we can easily check the following proposition:

Proposition 5.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $w \equiv 0$ and $\alpha_3 \neq 0$. Either of the following then occurs:*

(1) $\alpha_3 = -1/2$, $\alpha_4 = 1/2$ and

$$x \equiv \alpha_0 - \alpha_1, \quad y \equiv \frac{1}{2}, \quad z = t + (\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1), \quad w \equiv 0,$$

(2) $\alpha_3 + \alpha_4 = 0$, $(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1) = 0$ and

$$x \equiv \alpha_0 - \alpha_1, \quad y \equiv \frac{1}{2}, \quad z = -\frac{1}{2\alpha_3}t, \quad w \equiv 0.$$

Proposition 5.2 shows that if $w \equiv 0$ and $\alpha_3 \neq 0$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$,

$$s_3(x, y, z, w) = (\alpha_0 - \alpha_1, 1/2, \infty, 0).$$

Therefore, we have to consider the infinite solution such that $z \equiv \infty$, which is treated in the next section. If $w \equiv 0$ and $\alpha_3 = 0$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, we consider s_3 as the identical transformation.

In order to study π_2 , we assume that $x \equiv 0$ and obtain examples of the rational solutions.

Proposition 5.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $x \equiv 0$. Then, $\alpha_4 \neq 0$ and either of the following occurs:*

(1) $-\alpha_0 + \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$ and

$$x \equiv 0, \quad y \equiv \frac{1}{2}, \quad z = \frac{1}{2\alpha_4}t, \quad w \equiv 0,$$

(2) $\alpha_0 = \alpha_1 = 1/2$ and

$$x \equiv 0, y = \frac{1}{2} + \frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}, z = \frac{1}{2\alpha_4}t, w = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}.$$

Proposition 5.3 implies that if $x \equiv 0$ for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$,

$$\pi_2(x, y, z, w) = (2\alpha_4, 1/2, \infty, 0).$$

Therefore, we have to consider the infinite solution such that $z \equiv \infty$, which is treated in the next section.

6 Infinite solutions

In this section, we consider an “infinite solution,” that is, a solution such that some of (x, y, z, w) are identically equal to ∞ . Especially, we treat the infinite solution such that $z \equiv \infty$. For this purpose, following Sasano [28], we introduce the coordinate transformation which is given by

$$m_3 : x_3 = x, y_3 = y, z_3 = 1/z, w_3 = -(wz + \alpha_3)z,$$

By m_3 , we have the following proposition.

Proposition 6.1. *For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that $z \equiv \infty$. Either of the following then occurs:*

(1) $(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1) = 0$, $\alpha_4 = 0$, and

$$x_3 = \alpha_0 - \alpha_1, y_3 = \frac{1}{2}, z_3 = 0, w_3 = \frac{t}{2},$$

that is, $x = \alpha_0 - \alpha_1, y = 1/2, z = \infty, w = 0$,

(2) $\alpha_3 = 1/2, \alpha_4 = 0$.

$$x_3 = \alpha_0 - \alpha_1, y_3 = \frac{1}{2}, z_3 = 0, w_3 = \frac{t}{2} + \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{2},$$

that is, $x = \alpha_0 - \alpha_1, y = 1/2, z = \infty, w = 0$.

Proof. For $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, m_3 transforms the system of (x, y, z, w) into the system of (x_3, y_3, z_3, w_3) , which is given by

$$\begin{cases} tx'_3 = 2x_3^2y_3 - x_3^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x_3 - 2w_3 + t, \\ ty'_3 = -2x_3y_3^2 + 2x_3y_3 - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y_3 + \alpha_1, \\ tz'_3 = 2z_3^2w_3 + 1 - (1 - 2\alpha_3 - 2\alpha_4)z_3 - 2y_3 - tz_3^2, \\ tw'_3 = -2z_3w_3^2 + 2tz_3w_3 + \alpha_3t + (1 - 2\alpha_3 - 2\alpha_4)w_3. \end{cases}$$

Substituting $z_3 \equiv 0$ in

$$tz'_3 = 2z_3^2 w_3 + 1 - (1 - 2\alpha_3 - 2\alpha_4)z_3 - 2y_3 - tz_3^2,$$

we have $y_3 = 1/2$, which implies that $x_3 = \alpha_0 - \alpha_1$, because

$$ty'_3 = -2x_3 y_3^2 + 2x_3 y_3 - (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)y_3 + \alpha_1.$$

Substituting $x_3 = \alpha_0 - \alpha_1, y_3 = 1/2, z_3 = 0$ in

$$tx'_3 = 2x_3^2 y_3 - x_3^2 + (1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4)x_3 - 2w_3 + t,$$

we obtain

$$w_3 = \frac{t}{2} + \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{2}. \quad (6.1)$$

Substituting $z_3 \equiv 0$ and (6.1) in

$$tw'_3 = -2z_3 w_3^2 + 2tz_3 w_3 + \alpha_3 t + (1 - 2\alpha_3 - 2\alpha_4)w_3,$$

we have

$$0 = -\alpha_4 t + (1 - 2\alpha_3 - 2\alpha_4) \frac{(\alpha_0 - \alpha_1)(\alpha_0 + \alpha_1)}{2},$$

which proves the proposition. \square

Remark In both cases of Proposition 6.1, we can express the infinite solution by

$$x_3 = \alpha_0 - \alpha_1, y_3 = \frac{1}{2}, z_3 = 0, w_3 = \frac{t}{2} + \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{2}.$$

6.1 Bäcklund transformations and infinite solutions

In this subsection, we treat the relationship between the Bäcklund transformations and a solution such that $z \equiv \infty$ and $x, y, w \not\equiv \infty$. By m_3 and Proposition 6.1, we can prove the following proposition.

Proposition 6.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that $z \equiv \infty$ and $x, y, w \not\equiv \infty$. The actions of the Bäcklund transformations to the infinite solutions are then expressed by*

$$s_0 : (x, y, z, w) \longrightarrow (-\alpha_0 - \alpha_1, 1/2, \infty, 0),$$

$$s_1 : (x, y, z, w) \longrightarrow (\alpha_0 + \alpha_1, 1/2, \infty, 0),$$

$$s_2 : (x, y, z, w) \longrightarrow (\alpha_0 - \alpha_1, 1/2, \infty, 0),$$

$$s_3 : (x, y, z, w) \longrightarrow (\alpha_0 - \alpha_1, 1/2, t/\{2\alpha_3\} + (\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)/\{2\alpha_3\}, 0),$$

$$s_4 : (x, y, z, w) \longrightarrow (\alpha_0 - \alpha_1, 1/2, \infty, 0),$$

$$\pi_1 : (x, y, z, w) \longrightarrow (\alpha_1 - \alpha_0, 1/2, \infty, 0),$$

$$\pi_2 : (x, y, z, w) \longrightarrow (0, 1/2 + (\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)/\{2t\}, t/(\alpha_0 - \alpha_1), -(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)/\{2t\}).$$

Furthermore, if $\alpha_0 - \alpha_1 = 0$, π_2 is given by $(x, y, z, w) \longrightarrow (0, 1/2, \infty, 0)$.

Proof. We treat π_2 . The other cases can be proved in the same way. From the definitions of π_2 and m_3 , it follows that

$$\begin{aligned}\pi_2(x) &= \frac{t}{z} = tz_3 = 0, \\ \pi_2(y) &= -\frac{z}{t}(zw + \alpha_3) = \frac{w_3}{t} = \frac{1}{2} + \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{2t}, \\ \pi_2(z) &= \frac{t}{x} = \frac{t}{x_3} = \frac{t}{\alpha_0 - \alpha_1}, \\ \pi_2(w) &= -\frac{x}{t}(xy + \alpha_1) = -\frac{x_3}{t}(x_3y_3 + \alpha_1) = -\frac{(\alpha_0 - \alpha_1)(\alpha_0 + \alpha_1)}{2t}.\end{aligned}$$

If $\alpha_0 - \alpha_1 = 0$, by $m_3 \circ \pi_2 \circ m_3^{-1}$, we obtain

$$\begin{aligned}m_3(\pi_2(m_3^{-1}(x_3))) &= tz_3 = 0, \\ m_3(\pi_2(m_3^{-1}(y_3))) &= \frac{1}{2} + \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{2t} = \frac{1}{2}, \\ m_3(\pi_2(m_3^{-1}(z_3))) &= \frac{\alpha_0 - \alpha_1}{t} = 0, \\ m_3(\pi_2(m_3^{-1}(w_3))) &= -(\pi_2(w)\pi_2(z) + \pi_2(\alpha_3))\pi_2(z) = ty_3 = \frac{t}{2},\end{aligned}$$

which means that

$$\pi_2(x, y, z, w) = (0, 1/2, \infty, 0).$$

□

7 Necessary conditions ··· the case where z has a pole of order one at $t = \infty$

In this section, we assume that z has a pole of order one at $t = \infty$ and obtain the necessary conditions for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ to have such a rational solution.

7.1 The case where x, y, z, w are all holomorphic at $t = 0$

In this subsection, we assume that z has a pole of order one at $t = \infty$ and all of (x, y, z, w) are holomorphic at $t = 0$.

7.1.1 The case where $a_{0,0} = 0$

Proposition 7.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0$, $c_{0,0} \neq 0$. Then, $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$.*

Proof. From Proposition 3.11, it follows that $\alpha_0 - \alpha_1 \in \mathbb{Z}$. Furthermore, from Proposition 4.14, it follows that

$$h_{\infty,0} - h_{0,0} = 1/4 \cdot (\alpha_0 - \alpha_1)^2 + \alpha_4^2 - \alpha_4 \in \mathbb{Z}. \quad (7.1)$$

$s_4(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4)$ such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, for $s_4(x, y, z, w)$ x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0$, $c_{0,0} \neq 0$. From Proposition 4.14, it then follows that

$$h_{\infty,0} - h_{0,0} = 1/4 \cdot (\alpha_0 - \alpha_1)^2 + \alpha_4^2 + \alpha_4 \in \mathbb{Z}. \quad (7.2)$$

From (7.1) and (7.2), it follows that $2\alpha_4 \in \mathbb{Z}$. □

In order to treat the case where $a_{0,0} = c_{0,0} = 0$, let us prove the following two lemmas:

Lemma 7.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$ and $c_{0,0} = 0$. Then, $\alpha_4 \neq 0$ and $c_{0,1} = 1/\{2\alpha_4\}$.*

Proof. From Proposition 5.1, let us first note that $z \neq 0$. Suppose that $\alpha_4 = 0$. $s_4(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, 0)$ such that only $s_4(w)$ has a pole at $t = 0$, which is impossible from Proposition 2.1.

Thus, it follows that $\alpha_4 \neq 0$ and $s_4(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4)$ such that all of $s_4(x, y, z, w)$ are holomorphic at $t = 0$, which implies that $c_{0,1} = 1/\{2\alpha_4\}$. □

Lemma 7.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a solution such that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = c_{0,0} = 0$. Moreover, assume that $\alpha_2 \neq 0$. Then, $\alpha_4(\alpha_2 + \alpha_3 + \alpha_4) \neq 0$ and $a_{0,1} = (\alpha_3 + \alpha_4)/\{2\alpha_4(\alpha_2 + \alpha_3 + \alpha_4)\}$.*

Proof. From Proposition 5.1, let us first note that $x \neq z$. $s_2(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4)$ such that both of $s_2(y, w)$ have a pole at $t = 0$ and both of $s_2(x, z)$ are holomorphic at $t = 0$. Thus, it follows from Proposition 2.13 that $\alpha_2 + \alpha_3 + \alpha_4 \neq 0$ and

$$-\frac{\alpha_2}{a_{0,1} - 1/\{2\alpha_4\}} = \text{Res}_{t=0} s_2(y) = 2\alpha_4(\alpha_2 + \alpha_3 + \alpha_4),$$

which implies that $a_{0,1} = (\alpha_3 + \alpha_4)/2\alpha_4(\alpha_2 + \alpha_3 + \alpha_4)$.

□

By Lemmas 7.2 and 7.3, we can prove the following proposition:

Proposition 7.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0, c_{0,0} = 0$. One of the following then occurs: (1) $\alpha_0 = \alpha_1 = 0$, (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}, 2\alpha_4 = 1$, (3) $\alpha_0 - \alpha_1 \in \mathbb{Z}, 2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.*

Proof. From Proposition 3.11, it follows that $\alpha_0 - \alpha_1 \in \mathbb{Z}$. By Proposition 2.3 and 5.3, we can assume that $x \not\equiv 0$ and

$$a_{0,0} = 0, b_{0,0} = \alpha_1/(\alpha_0 + \alpha_1), c_{0,0} = 0, d_{0,0} = -\alpha_3(\alpha_0 - \alpha_1)/\{(2\alpha_4 - 1)(\alpha_0 + \alpha_1)\},$$

where $\alpha_0 + \alpha_1 \neq 0$ and $2\alpha_4 \neq 1$.

If $\alpha_0 - \alpha_1 \neq 0$ and $\alpha_3 \neq 0$, then $s_3(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, w)$ are holomorphic at $t = \infty$. Moreover, all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$ and for $s_3(x, y, z, w)$, $a_{0,0} = 0, c_{0,0} \neq 0$. Thus, by Proposition 7.1, we find that $\alpha_0 - \alpha_1 \in \mathbb{Z}, 2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.

If $\alpha_0 - \alpha_1 \neq 0$ and $\alpha_3 = 0$, then $s_4(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2, 2\alpha_4, -\alpha_4)$ such that $s_4(z)$ has a pole of order one at $t = \infty$ and all of $s_4(x, y, w)$ are holomorphic at $t = \infty$. Moreover, $s_4(x, y, z, w)$ are all holomorphic at $t = 0$ and for $s_4(x, y, z, w)$, $a_{0,0} = 0, c_{0,0} = 0$. Thus, from the above discussion, we see that $\alpha_0 - \alpha_1 \in \mathbb{Z}, 2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.

Let us suppose that $\alpha_0 - \alpha_1 = 0$ and $\alpha_2 \neq 0$. We can then assume that $\alpha_3 + \alpha_4 \neq 0$. Thus, $\pi_2(x, y, z, w)$ is a solution of $B_4^{(1)}(2\alpha_4 + \alpha_3, \alpha_3, \alpha_2, \alpha_1, 0)$ such that $\pi_2(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $\pi_2(x, y, w)$ are holomorphic at $t = \infty$. Moreover, all of $\pi_2(x, y, z, w)$ are holomorphic at $t = 0$ and for $\pi_2(x, y, z, w)$, $a_{0,0} = (2\alpha_4 + \alpha_3) - \alpha_3 \neq 0$, $b_{0,0} = -\alpha_3/\{2\alpha_4\}$, $c_{0,0} = (\alpha_2 + \alpha_3 + \alpha_4)/(\alpha_3 + \alpha_4)$, which implies that $\alpha_0 - \alpha_1 = 0$ and $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$ from Proposition 8.2.

Let us consider that $\alpha_0 - \alpha_1 = 0$ and $\alpha_2 = 0$. $s_1 s_0(x, y, z, w)$ is then a solution of $B_4^{(1)}(-\alpha_0, -\alpha_1, 2\alpha_0, \alpha_3, \alpha_4)$ such that only $s_1 s_0(z)$ has a pole of order one at $t = \infty$. Moreover, all of $s_1 s_0(x, y, z, w)$ are holomorphic at $t = 0$ and for $s_1 s_0(x, y, z, w)$, $a_{0,0} = 0, b_{0,0} = 1/2, c_{0,0} = 0$. Thus, it follows from the above discussion that $\alpha_0 - \alpha_1 = 0$ and $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.

□

Remark Proposition 8.2 can be proved independent of the propositions in this section.

Let us summarize Propositions 7.1 and 7.4.

Proposition 7.5. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0$. One of the following then occurs:*

- (1) $\alpha_0 = \alpha_1 = 0$,
- (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (3) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.

7.1.2 $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$

We treat the case where x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} \neq 0$. Let us first deal with the case in which $b_{0,0} \neq 1/2$.

Proposition 7.6. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1) \neq 1/2$. Either of the following then occurs:*

- (1) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (2) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.

Proof. Since $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, it follows that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. If $\alpha_0 \neq 0$, then using s_0 we find that $s_0(x, y, z, w)$ is a solution of $B_4^{(1)}(-\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4)$ such that only $s_0(z)$ has a pole of order one at $t = 0$ and all of $S_0(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_0(x, y, z, w)$ $a_{0,0} = 0$. Thus, the proposition follows from Proposition 7.5.

If $\alpha_1 \neq 0$, we use s_1 in the same way and can obtain the necessary conditions. □

Let us treat the case where $a_{0,0} \neq 0$ and $b_{0,0} = 1/2$.

Proposition 7.7. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, $b_{0,0} = 1/2$. One of the following then occurs:*

- (1) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (2) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (3) $\alpha_0 + \alpha_1 + 2\alpha_2 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$.

Proof. By Proposition 2.5, we can assume that $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$, $c_{0,0} = 0$, or that $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$, $c_{0,0} = (\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)/(1 - 2\alpha_3 - 2\alpha_4) \neq 0$.

Let us first suppose that $\alpha_2 \neq 0$. $s_2(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4)$ such that only $s_2(z)$ has a pole of order one at $t = \infty$ and all of

$s_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_2(x, y, z, w)$, $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$ and $b_{0,0} \neq 1/2$. Thus, from Proposition 7.6, we can obtain the necessary conditions.

Now, let us consider the case where $\alpha_2 = 0$. Since $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, it follows that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. We suppose that $\alpha_0 \neq 0$. $s_0(x, y, z, w)$ is then a solution of $B_4^{(1)}(-\alpha_0, \alpha_1, \alpha_0, \alpha_3, \alpha_4)$ such that only $s_2(z)$ has a pole of order one at $t = \infty$ and all of $s_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_0(x, y, z, w)$, $a_{0,0} = -\alpha_0 - \alpha_1$ and $b_{0,0} \neq 1/2$. When $\alpha_0 + \alpha_1 = 0$, we can obtain the necessary conditions from Proposition 7.5. When $\alpha_0 + \alpha_1 \neq 0$, we can obtain the necessary conditions from the above discussion.

If $\alpha_2 = 0$ and $\alpha_1 \neq 0$, using s_1 in the same way, we can obtain the necessary conditions. \square

7.2 The case where z has a pole at $t = 0$

Proposition 7.8. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that z has a pole at $t = 0$ and x, y, w are all holomorphic at $t = 0$. One of the following then occurs:*

- (1) $\alpha_0 = \alpha_1 = 0$,
- (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (3) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (4) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (5) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (6) $2\alpha_3 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$.

Proof. Let us first suppose that $\alpha_3(\alpha_3 + \alpha_4) \neq 0$. $s_3(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that only $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_3(x, y, z, w)$, $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$, $d_{0,0} = 0$. If $\alpha_0 - \alpha_1 = 0$, we can obtain the necessary conditions from Proposition 7.5. If $\alpha_0 - \alpha_1 \neq 0$, we can obtain the necessary conditions from Proposition 7.6 and 7.7.

Now, let us consider the case where $\alpha_3 \neq 0$ and $\alpha_3 + \alpha_4 = 0$. It then follows from Proposition 5.2 that $w \neq 0$. Thus, $s_3(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that only $s_3(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_3(x, y, z, w)$, $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$, $d_{0,0} = 0$. Therefore, using Proposition 8.3, we can obtain the necessary conditions.

If $\alpha_3 = 0$, then $\alpha_3 + \alpha_4 \neq 0$, because $\alpha_4 \neq 0$. Thus, $s_4(x, y, z, w)$ is a solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2, 2\alpha_4, -\alpha_4)$ such that only $s_4(z)$ has a pole of order one at $t = \infty$ and has a pole at $t = 0$. Therefore, we can obtain the necessary condition based on the above discussion. \square

7.3 The case where y, w have a pole at $t = 0$

Proposition 7.9. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that y, w both have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$. One of the following then occurs:*

- (1) $\alpha_0 + \alpha_2 = \alpha_1 + \alpha_2 = 0$,
- (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (3) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $\alpha_0 + \alpha_1 \in \mathbb{Z}$,
- (4) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (5) $2\alpha_3 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$.

Proof. By Proposition 2.13, let us note that $\alpha_4(\alpha_3 + \alpha_4) \neq 0$. We first assume that $\alpha_2 \neq 0$. $s_2(x, y, z, w)$ is then a solution of $B_4^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4)$ such that only $s_2(z)$ has a pole of order one at $t = \infty$ and all of $s_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_2(x, y, z, w)$, $a_{0,0} = c_{0,0} = 0$. Thus, by Proposition 7.4, we obtain the necessary conditions.

Now, let us suppose that $\alpha_2 = 0$ and $\alpha_0 \neq 0$. $s_0(x, y, z, w)$ is then a solution of $B_4^{(1)}(-\alpha_0, \alpha_1, \alpha_0, \alpha_3, \alpha_4)$ such that only $s_2(z)$ has a pole of order one at $t = \infty$ and only $s_2(y, w)$ have a pole at $t = 0$. Based on the above discussion, we then obtain the necessary condition.

If $\alpha_2 = 0$ and $\alpha_1 \neq 0$, using s_1 in the same way, we obtain the necessary conditions.

If $\alpha_0 = \alpha_1 = \alpha_2 = 0$, the parameters then satisfy one of the conditions in the proposition. \square

8 Necessary conditions · · · the case where z has a pole of order n ($n \geq 2$) at $t = \infty$

In this section, for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, we treat a rational solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and assume that $\alpha_4 = 0$. This section consists of three subsections. In the first, second and third subsections, we treat the case where x, y, z, w are all holomorphic at $t = 0$, the case where z has a pole at $t = 0$, and the case where y, w both have a pole at $t = 0$, respectively.

8.1 The case where x, y, z, w are all holomorphic at $t = 0$

8.1.1 The case where $a_{0,0} = 0$

Proposition 8.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $\alpha_4 = 0$ and there exists a rational solution such that z has a pole of order ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0$. Then, $\alpha_4 = 0$, $\alpha_0 - \alpha_1 \in \mathbb{Z}$.*

Proof. This proposition follows from Proposition 3.11. □

8.1.2 $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$

Let us first treat the case where x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} \neq 0$, $b_{0,0} \neq 1/2$.

Proposition 8.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $\alpha_4 = 0$ and there exists a rational solution such that z has a pole of order ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$ and $b_{0,0} = -\alpha_1/(\alpha_0 - \alpha_1) \neq 1/2$. Then, $\alpha_4 = 0$, $\alpha_0 + \alpha_1 \in \mathbb{Z}$.*

Proof. Since $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$, it follows that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. If $\alpha_0 \neq 0$ by s_0 and Proposition 8.1, we can prove the proposition.

If $\alpha_1 \neq 0$, using s_0 in the same way, we can show the proposition. □

Let us treat the case where x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} \neq 0$, $b_{0,0} = 1/2$.

Proposition 8.3. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $\alpha_4 = 0$ and there exists a rational solution such that z has a pole of order ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = \alpha_0 - \alpha_1 \neq 0$ and $b_{0,0} = 1/2$. Either of the following then occurs:*

- (1) $\alpha_4 = 0$, $\alpha_0 + \alpha_1 = 0$,
- (2) $\alpha_4 = 0$, $\alpha_0 + \alpha_1 + 2\alpha_2 \in \mathbb{Z}$.

Proof. By Proposition 2.5, we can assume that

$$a_{0,0} = \alpha_0 - \alpha_1, \quad b_{0,0} = \frac{1}{2}, \quad c_{0,0} = \frac{(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)}{1 - 2\alpha_3 - 2\alpha_4} \neq 0, \quad d_{0,0} = -\frac{(1 - 2\alpha_4)(1 - 2\alpha_3 - 2\alpha_4)}{2(\alpha_0 + \alpha_1)(\alpha_0 - \alpha_1)},$$

where $\alpha_3 + \alpha_4 \neq 1/2$.

Suppose that $\alpha_2 \neq 0$. $s_2(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(\alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4)$ such that $s_2(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $s_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} \neq 1/2$. Thus, by Proposition 8.2, we can obtain the necessary conditions.

Suppose that $\alpha_2 = 0$. Since $\alpha_0 - \alpha_1 \neq 0$, it follows that $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$. Let us first assume that $\alpha_0 \neq 0$. $s_0(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(-\alpha_0, \alpha_1, \alpha_0, \alpha_3, \alpha_4)$ such that $s_0(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $s_0(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, assume that $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$. Based on the above discussion, we can obtain the necessary conditions.

If $\alpha_2 = 0$ and $\alpha_1 \neq 0$, we use s_1 in the same way and can obtain the necessary conditions. \square

8.2 The case where z has a pole of order $m \geq 1$ at $t = 0$

Proposition 8.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, $\alpha_4 = 0$ and there exists a rational solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that z has a pole of order $m \geq 1$ at $t = 0$. One of the following then occurs:*

- (1) $\alpha_0 - \alpha_1 = 0$, $\alpha_4 = 0$,
- (2) $2\alpha_3 \in \mathbb{Z}$, $\alpha_4 = 0$,
- (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $\alpha_4 = 0$,
- (4) $2\alpha_3 \in \mathbb{Z}$, $\alpha_4 = 0$.

Proof. We may assume that $\alpha_3 \neq 0$. It then follows from Corollaries 1.6 and 2.8 that $s_3(x, y, z, w)$ is a rational solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ such that only $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_3(x, y, z, w)$, $a_{0,0} = \alpha_0 - \alpha_1$, $b_{0,0} = 1/2$, $d_{0,0} = 0$. We can now assume that $\alpha_0 - \alpha_1 \neq 0$. Thus, by Proposition 7.7, we can obtain the necessary conditions. \square

8.3 The case where y, w have a pole at $t = 0$

Proposition 8.5. *If $\alpha_4 = 0$, then for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no rational solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$ and y, w have a pole at $t = 0$ and x, z are both holomorphic at $t = 0$.*

Proof. This proposition follows from Proposition 2.13. \square

9 The standard forms of the parameters for rational solutions

Let us summarize the discussion in Section 7, 8.

Proposition 9.1. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. One of the following then occurs:*

- (1) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$,
- (4) $\alpha_0 + \alpha_1 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$,
- (5) $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $\alpha_0 + \alpha_1 \in \mathbb{Z}$,
- (6) $2\alpha_3 \in \mathbb{Z}$, $2\alpha_4 \in \mathbb{Z}$.

Corollary 9.2. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters can then be transformed so that $\alpha_0 - \alpha_1 \in \mathbb{Z}$, $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$.*

9.1 Shift operators

Following Sasano [27], we obtain shift operators of the parameters.

Proposition 9.3. *T_1, T_2, T_3 and T_4 are defined by*

$$T_1 = s_4 \pi_1 s_1 s_2 s_4 s_3 s_4 s_3 s_2 s_1, \quad T_2 = s_0 T_1 s_0, \quad T_3 = s_2 T_2 s_2, \quad T_4 = s_3 T_3 s_3,$$

respectively. Then,

$$\begin{aligned} T_1(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, -1, 0, 0, 0), \\ T_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (-1, -1, 1, 0, 0), \\ T_3(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, -1, 1, 0), \\ T_4(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, 0, -1, 1). \end{aligned}$$

9.2 Standard forms of the parameters for rational solutions

Proposition 9.4. *Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs: (1) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$, (2) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$. The cases (1) and (2) denote the standard forms I and II, respectively.*

Especially, the parameters can be transformed into the standard form I, if they satisfy one of the conditions in our main theorem.

Proof. By Corollary 9.2, we may assume that $\alpha_0 - \alpha_1 \in \mathbb{Z}$ and $2\alpha_3 + 2\alpha_4 \in \mathbb{Z}$. One of the following cases then occurs:

- (1) $\alpha_0 - \alpha_1 \equiv 2\alpha_3 + 2\alpha_4 \equiv 0 \pmod{2}$, (2) $\alpha_0 - \alpha_1 \equiv 0, 2\alpha_3 + 2\alpha_4 \equiv 1 \pmod{2}$,
- (3) $\alpha_0 - \alpha_1 \equiv 1, 2\alpha_3 + 2\alpha_4 \equiv 0 \pmod{2}$, (4) $\alpha_0 - \alpha_1 \equiv 2\alpha_3 + 2\alpha_4 \equiv 1 \pmod{2}$.

If case (1) occurs, using T_2, T_3 , we have $\alpha_0 - \alpha_1 = \alpha_3 + \alpha_4 = 0$.

If case (2) occurs, using T_2, T_3 , we obtain $\alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2$.

If case (3) occurs, using T_2, T_3 , we have $\alpha_0 - \alpha_1 = 1, \alpha_3 + \alpha_4 = 0$. Furthermore, by $\pi_2 s_3 s_1$, we obtain $\alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2$.

If case (4) occurs, using T_2, T_3 , we have $\alpha_0 - \alpha_1 = \alpha_3 + \alpha_4 = 1$, which implies that

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-\alpha_2 + 1/2, -\alpha_2 - 1/2, \alpha_2, \alpha_3, -\alpha_3 + 1/2).$$

By $s_2 s_1 s_2 T_3$, we obtain

$$(-\alpha_2 + 1/2, -\alpha_2 - 1/2, \alpha_2, \alpha_3, -\alpha_3 + 1/2) \longrightarrow (-\alpha_2, -\alpha_2, \alpha_2 - 1/2, \alpha_3 + 1/2, -\alpha_3 + 1/2).$$

If $\alpha_0 - \alpha_1 = \alpha_3 + \alpha_4 = 0$, by T_4 , we may assume that $\alpha_0 - \alpha_1 = \alpha_3 + \alpha_4 = 0$ and $\alpha_4 \neq 0$.

□

If $\alpha_0 - \alpha_1 = 0$ and $\alpha_3 + \alpha_4 = 0$, it follows from Corollary 1.23 that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a unique rational solution which is expressed by

$$x \equiv 0, y \equiv 1/2, z = 1/\{2\alpha_4\} \cdot t, w \equiv 0.$$

We have then to only treat the standard form II in order to classify the rational solutions of $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$.

10 The standard form II \cdots (1)

10.1 The case where z has a pole of order n ($n \geq 2$) at $t = \infty$

By Corollary 1.7, we can prove the following proposition:

Proposition 10.1. *Suppose that $\alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. Moreover, assume that z has a pole of order n ($n \geq 2$) at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. One of the following then occurs:*

- (1) $\alpha_0 - \alpha_1 = 0, \alpha_3 = 1/2, \alpha_4 = 0$,
- (2) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1/2$.

10.2 The case where z has a pole of order one at $t = \infty$

Proposition 10.2. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. One of the following then occurs:*

- (1) x, y, z, w are all holomorphic at $t = 0$ and $a_{0,0} = 0$, $c_{0,0} \neq 0$,
- (2) z has a pole at $t = 0$ and x, y, w are all holomorphic at $t = 0$,
- (3) y, w both have a pole of order one at $t = 0$ and x, z are holomorphic at $t = 0$.

Proof. If x, y, z, w are all holomorphic at $t = 0$, it follows from Proposition 2.2 that $a_{0,0} = 0$, because $\alpha_0 - \alpha_1 = 0$. Moreover,

$$h_{0,0} = \begin{cases} 0, & \text{if } c_{0,0} = 0, \\ \alpha_3(\alpha_3 + 2\alpha_4 - 1), & \text{if } c_{0,0} \neq 0. \end{cases}$$

On the other hand, $h_{\infty,0} = -1/4$. Thus, $c_{0,0} \neq 0$ because $h_{\infty,0} - h_{0,0} \in \mathbb{Z}$. □

10.2.1 The case where x, y, z, w are all holomorphic at $t = 0$

By Proposition 7.1, we have the following proposition:

Proposition 10.3. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$. Then, $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $2\alpha_4 \in \mathbb{Z}$.*

10.2.2 The case where z has a pole at $t = 0$

Proposition 10.4. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that z has a pole at $t = 0$ and x, y, w are all holomorphic at $t = 0$. One of the following then occurs:*

- (1) $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$,
- (2) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $\alpha_4 \in \mathbb{Z}$.

Proof. We may assume that $\alpha_0 = \alpha_1 \neq 0$ and $\alpha_3 \neq 1/2$. Moreover, from Propositions 4.1 and 4.3, it follows that $h_{\infty,0} - h_{0,0} = \alpha_4^2 - \alpha_4 \in \mathbb{Z}$, which implies that $\alpha_3 \notin \mathbb{Z}$.

$s_3(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, 1/2)$ such that $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$ and $a_{0,0} = 0$. If $c_{0,0} \neq 0$ for $s_3(x, y, z, w)$, it follows from Propositions 4.1 and 4.2

that $h_{\infty,0} - h_{0,0} = -1/4 \in \mathbb{Z}$, which is impossible. Thus, we have $a_{0,0} = c_{0,0} = 0$ for $s_3(x, y, z, w)$.

Now, let us assume that $\alpha_2 + \alpha_3 \neq 0$. It then follows from Lemmas 7.2 and 7.3 that $s_3\pi_2(x, y, z, w)$ is a rational solution of $B_4^{(1)}(1 - \alpha_3, -\alpha_3, \alpha_2 + \alpha_3, \alpha_1, 0)$ such that $s_3\pi_2(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $s_3\pi_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $s_3\pi_2(x, y, z, w)$, $a_{0,0} = 1$, $b_{0,0} = \alpha_3$, $c_{0,0} \neq 0$. Thus, by Proposition 8.2, we find that $1 - 2\alpha_3 \in \mathbb{Z}$, which implies that $\alpha_4 \in \mathbb{Z}$ because $\alpha_3 + \alpha_4 = 1/2$ and $\alpha_3 \notin \mathbb{Z}$.

Let us treat the case where $\alpha_2 + \alpha_3 = 0$. $s_0s_1s_3(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(-\alpha_0, -\alpha_1, 2\alpha_0, -\alpha_3, 1/2)$ such that $s_0s_1s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_1s_0s_3(x, y, z, w)$ are holomorphic at $t = 0$. Furthermore, for $s_0s_1s_3(x, y, z, w)$, $a_{0,0} = c_{0,0} = 0$. Thus, by π_2 , we have $\alpha_4 \in \mathbb{Z}$. \square

10.2.3 The case where y, w both have a pole of order one at $t = 0$

Proposition 10.5. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that y, w both have a pole of order one at $t = 0$ and x, z are both holomorphic at $t = 0$. Then, $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $\alpha_0 + \alpha_1 \in \mathbb{Z}$.*

Proof. We may assume that $\alpha_2 + \alpha_3 + \alpha_4 \neq 0$. Moreover, by Proposition 5.3, we can suppose that $x \neq 0$.

$\pi_2(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(2\alpha_4 + \alpha_3, \alpha_3, \alpha_2, \alpha_1, 0)$ such that $\pi_2(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $\pi_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $\pi_2(x, y, z, w)$, $a_{0,0} = 2\alpha_4$, $b_{0,0} = 1/2$. By Proposition 8.3, we then obtain the necessary condition. \square

10.3 Summary

Proposition 10.6. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. One of the following then occurs:*

- (1) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $2\alpha_4 \in \mathbb{Z}$,
- (2) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $\alpha_0 + \alpha_1 \in \mathbb{Z}$.

Corollary 10.7. *Suppose that $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

- (1) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$,
- (2) $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$.

Especially, the parameters can be transformed into those of the standard form I if they satisfy one of the following:

- (i) $\alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2, \alpha_4 \in \mathbb{Z},$
- (ii) $\alpha_0 - \alpha_1 = 0, \alpha_3 + \alpha_4 = 1/2, \alpha_0 + \alpha_1 \in \mathbb{Z}, \alpha_0 + \alpha_1 \equiv 1 \pmod{2}.$

If the above cases do not occur, the parameters can be transformed into those of (2).

11 Standard forms II... (2)

In this section, we treat the case where $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2$.

11.1 The case where z has a pole of order n ($n \geq 2$) at $t = \infty$

By Proposition 1.7, we can first prove the following proposition:

Proposition 11.1. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order n ($n \geq 2$) at $t = \infty$ and all of x, y, w are holomorphic at $t = \infty$. One of the following then occurs:*

- (1) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = 1/2,$
- (2) $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 = 1/2, \alpha_4 = 0.$

11.2 The case where z has a pole of order one at $t = \infty$

In order to treat the case where z has a pole of order one at $t = \infty$, we prove the following lemma:

Lemma 11.2. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that all of x, y, w are holomorphic at $t = \infty$. One of the following then occurs:*

- (1) x, y, z, w are all holomorphic at $t = 0$ and $c_{0,0} \neq 0,$
- (2) only z has a pole at $t = 0$.

Proof. From Proposition 4.1, we find that $h_{\infty,0} = -1/4$. Thus, the lemma follows from Propositions 4.2, 4.3, 4.4 and 4.14. □

11.2.1 The case where x, y, z, w are all holomorphic at $t = 0$

By Proposition 7.1, we obtain the following proposition:

Proposition 11.3. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that x, y, z, w are all holomorphic at $t = 0$ and $c_{0,0} \neq 0$. Then, $\alpha_0 = \alpha_1 = \alpha_2 = 0, \alpha_3 + \alpha_4 = 1/2, 2\alpha_4 \in \mathbb{Z}.$*

11.2.2 The case where z has a pole at $t = 0$

Proposition 11.4. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution such that z has a pole of order one at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$. Moreover, assume that only z has a pole at $t = 0$. Then, $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $\alpha_4 \in \mathbb{Z}$.*

Proof. We may assume that $\alpha_3 \neq 1/2$. Moreover, from Propositions 4.1 and 4.3, it follows that $h_{\infty,0} - h_{0,0} = \alpha_4^2 - \alpha_4 \in \mathbb{Z}$, which implies that $\alpha_3 \notin \mathbb{Z}$ because $\alpha_3 + \alpha_4 = 1/2$.

$s_3(x, y, z, w)$ is then a rational solution of $B_4^{(1)}(0, 0, \alpha_3, -\alpha_3, 1/2)$ such that $s_3(z)$ has a pole of order one at $t = \infty$ and all of $s_3(x, y, z, w)$ are holomorphic at $t = 0$. Furthermore, for $s_3(x, y, z, w)$, $a_{0,0} = 0$. If $c_{0,0} \neq 0$ for $s_3(x, y, z, w)$, it follows from Propositions 4.1 and 4.2 that $h_{\infty,0} - h_{0,0} = -1/4 \in \mathbb{Z}$, which is impossible. Thus, for $s_3(x, y, z, w)$, $a_{0,0} = c_{0,0} = 0$. Therefore, from Lemmas 7.2 and 7.3, we find that $s_3\pi_2(x, y, z, w)$ is a rational solution of $B_4^{(1)}(1 - \alpha_3, -\alpha_3, \alpha_3, 0, 0)$ such that $s_3\pi_2(z)$ has a pole of order n ($n \geq 2$) at $t = \infty$ and all of $s_3\pi_2(x, y, z, w)$ are holomorphic at $t = 0$. Moreover, for $\pi_2 s_3(x, y, z, w)$, $a_{0,0} = 1, b_{0,0} = \alpha_3$, which implies that $1 - 2\alpha_3 \in \mathbb{Z}$, that is, $\alpha_4 \in \mathbb{Z}$ from Proposition 8.2. \square

11.3 Summary

Let us summarize the results in this section.

Proposition 11.5. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. Then, $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $2\alpha_4 \in \mathbb{Z}$.*

Corollary 11.6. *Suppose that $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$ and for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters can then be transformed so that one of the following occurs:*

- (1) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$,
- (2) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1/2$.

Especially, the parameters can be transformed into the standard form I, if $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_3 + \alpha_4 = 1/2$, $\alpha_4 \in \mathbb{Z}$. Otherwise, the parameters can be transformed into those of case (2).

12 Standard form II. . . (3)

In this section, we treat the case where $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1/2$. For this purpose, let us first prove the following lemma:

Lemma 12.1. *Suppose that for $B_4^{(1)}(0, 0, 0, 0, 1/2)$, there exists a rational solution. z then has a pole of order n ($n \geq 2$) at $t = \infty$.*

Proof. Suppose that z has a pole of order one at $t = \infty$. By Proposition 4.1, we then see that $h_{\infty,0} = -1/4$.

On the other hand, by Propositions 4.2, 4.3, and 4.4, we find that $h_{0,0} = 0$. Thus, it follows that $h_{\infty,0} - h_{0,0} = -1/4 \notin \mathbb{Z}$, which contradicts Proposition 4.14. \square

We can then prove the following proposition:

Proposition 12.2. *For $B_4^{(1)}(0, 0, 0, 0, 1/2)$, there exists no rational solution.*

Proof. Suppose that for $B_4^{(1)}(0, 0, 0, 0, 1/2)$, there exists a rational solution. By Lemma 12.1, we can then assume that z has a pole of order n ($n \geq 2$) at $t = \infty$.

$s_4 s_3(x, y, z, w)$ is a rational solution of $B_4^{(1)}(0, 0, 1, -1, 1/2)$ such that z has a pole of order one at $t = \infty$. It then follows that for $s_4 s_3(x, y, z, w)$, $h_{\infty,0} = 3/4$. On the other hand, for $s_4 s_3(x, y, z, w)$, $h_{0,0} \in \mathbb{Z}$. Thus, it follows that $h_{\infty,0} - h_{0,0} = 3/4 \notin \mathbb{Z}$, which contradicts Proposition 4.14. \square

13 Proof of main theorem

In this section, we prove the main theorem.

Proof. Suppose that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ there exists a rational solution. From Proposition 9.4, and Corollaries 10.7 and 11.6, it follows that the parameters can be transformed so that one of the following occurs:

(1) $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$, (2) $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1/2$.

Especially, from Proposition 9.4, and Corollaries 10.7 and 11.6, we see that the parameters can be transformed into the standard form I if they satisfy one of the conditions in this theorem. Otherwise, the parameters can be transformed into those of (2).

If $\alpha_0 - \alpha_1 = 0$, $\alpha_3 + \alpha_4 = 0$, $\alpha_4 \neq 0$, then from Corollary 1.23, we observe that for $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ there exists a rational solution and $x \equiv 0$, $y \equiv 1/2$, $z = 1/\{2\alpha_4\} \cdot t$, $w \equiv 0$ and it is unique.

From Proposition 12.2, we see that for $B_4^{(1)}(0, 0, 0, 0, 1/2)$, there exists no rational solution. \square

A Rational solutions of the Sasano system of type $D_4^{(1)}$

Following Sasano [28], we introduce the Sasano system of type $D_4^{(1)}$, which is defined by

$$D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4} \begin{cases} tx' = 2x^2y - x^2 + (\alpha_0 + \alpha_1)x - 2w + t, \\ ty' = -2xy^2 + 2xy - (\alpha_0 + \alpha_1)y + \alpha_1, \\ tz' = 2z^2w - tz^2 - (1 - \alpha_3 - \alpha_4)z + 1 - 2y, \\ tw' = -2zw^2 + 2tzw + (1 - \alpha_3 - \alpha_4)w + \alpha_3t, \\ \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1. \end{cases}$$

$D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has the Bäcklund transformations, $s_0, s_1, s_2, s_3, s_4, \pi_1, \pi_2, \pi_3, \pi_4$, which are given by

$$\begin{aligned} s_0 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x + \frac{\alpha_0}{y-1}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4\right), \\ s_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4\right), \\ s_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \\ &\left(x, y - \frac{\alpha_2 z}{xz-1}, z, w - \frac{\alpha_2 x}{xz-1}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2\right), \\ s_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4\right), \\ s_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(x, y, z + \frac{\alpha_4}{w-t}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4\right), \\ \pi_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (-x, 1-y, -z, -w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4), \\ \pi_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x, y, z, w-t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_4, \alpha_3), \\ \pi_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(tz, \frac{w}{t}, \frac{x}{t}, ty, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right), \\ \pi_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow \left(-tz, \frac{t-w}{t}, -\frac{x}{t}, t-ty, t; \alpha_3, \alpha_4, \alpha_2, \alpha_0, \alpha_1\right). \end{aligned}$$

The Bäcklund transformation group $\langle s_0, s_1, s_2, s_3, s_4, \pi_1, \pi_2, \pi_3, \pi_4 \rangle$ is isomorphic to $\tilde{W}(D_4^{(1)})$.

A.1 The properties of the Bäcklund transformations

By direct calculation, we obtain the following proposition:

Proposition A.1. (0) If $y \equiv 1$ for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_0 = 0$.

- (1) If $y \equiv 0$ for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_1 = 0$.
- (2) If $xz \equiv 1$ for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_2 = 0$.
- (3) If $w \equiv 0$ for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_3 = 0$.
- (4) If $w = t$ for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_4 = 0$.

By Proposition A.1, we do not have to consider the infinite solutions of $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$.

A.2 Main theorem for $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$

Sasano [28] proved that $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ is equivalent to $B_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$.

Proposition A.2. Suppose that (x, y, z, w) is a solution of $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ and

$$\begin{aligned} X = x, Y = y, Z = \frac{1}{z}, W = -(zw + \alpha_3)z, \\ A_0 = \alpha_0, A_1 = \alpha_1, A_2 = \alpha_2, A_3 = \alpha_3, A_4 = \frac{\alpha_4 - \alpha_3}{2}. \end{aligned}$$

(X, Y, Z, W) is then a solution of $B_4^{(1)}(A_j)_{0 \leq j \leq 4}$.

Theorem A.3. Suppose that $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution. By some Bäcklund transformations, the parameters and solution can then be transformed so that

$$\alpha_0 - \alpha_1 = \alpha_3 + \alpha_4 = 0, \text{ and } (x, y, z, w) = (0, 1/2, 2\alpha_4/t, t/2).$$

Moreover, $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution if and only if one of the following occurs:

- (1) $\alpha_0 - \alpha_1 \in \mathbb{Z}, \quad \alpha_3 + \alpha_4 \in \mathbb{Z}, \quad \alpha_0 - \alpha_1 \equiv \alpha_3 + \alpha_4 \pmod{2},$
- (2) $\alpha_0 - \alpha_1 \in \mathbb{Z}, \quad \alpha_3 - \alpha_4 \in \mathbb{Z}, \quad \alpha_0 - \alpha_1 \equiv \alpha_3 - \alpha_4 \pmod{2},$
- (3) $\alpha_0 + \alpha_1 \in \mathbb{Z}, \quad \alpha_3 + \alpha_4 \in \mathbb{Z}, \quad \alpha_0 + \alpha_1 \equiv \alpha_3 + \alpha_4 \pmod{2},$
- (4) $\alpha_0 + \alpha_1 \in \mathbb{Z}, \quad \alpha_3 - \alpha_4 \in \mathbb{Z}, \quad \alpha_0 + \alpha_1 \equiv \alpha_3 - \alpha_4 \pmod{2},$
- (5) $\alpha_0 - \alpha_1 \in \mathbb{Z}, \quad \alpha_0 + \alpha_1 \in \mathbb{Z}, \quad \alpha_0 - \alpha_1 \not\equiv \alpha_0 + \alpha_1 \pmod{2},$
- (6) $\alpha_3 - \alpha_4 \in \mathbb{Z}, \quad \alpha_3 + \alpha_4 \in \mathbb{Z}, \quad \alpha_3 - \alpha_4 \not\equiv \alpha_3 + \alpha_4 \pmod{2}.$

Remark In Theorem A.3, we can assume that $\alpha_1, \alpha_4 \neq 0$. For this purpose, following Sasano [28], we define

$$\begin{aligned} T_1 &:= s_3 s_0 s_2 s_4 s_1 s_2 \pi_4, & T_2 &:= s_4 s_1 s_2 s_3 s_0 s_2 \pi_4, \\ T_3 &:= s_3 s_2 s_0 s_1 s_2 s_3 \pi_1 \pi_2, & T_4 &:= s_4 s_3 s_2 s_1 s_0 s_2 \pi_1 \pi_2, \end{aligned}$$

which implies that

$$\begin{aligned} T_1(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, 0, -1, 1, 0), \\ T_2(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 1, -1, 0, 1), \\ T_3(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, 0, 1, -1), \\ T_4(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, -1, 1, 1). \end{aligned}$$

By T_3 , we may suppose that $\alpha_4 \neq 0$. By $T_1 T_2 T_4^{-1}$, we can assume that $\alpha_1 \neq 0$.

B Rational solutions of the Sasano system of type $D_5^{(2)}$

$$D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4} \begin{cases} tx' = 2x^2y - tx^2 - 2\alpha_0x + 1 - 2x^2z(zw + \alpha_3), \\ ty' = -2xy^2 + 2txy + 2\alpha_0y + \alpha_1t + 2z(zw + \alpha_3)(2xy + \alpha_1), \\ tz' = 2z^2w - z^2 + (1 - 2\alpha_4)z + t - 2xz^2(xy + \alpha_1), \\ tw' = -2zw^2 + 2zw - (1 - 2\alpha_4)w + \alpha_3 + 2x(xy + \alpha_1)(2zw + \alpha_3), \\ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1/2. \end{cases}$$

$$\begin{aligned} s_0 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(-x, -y + \frac{2\alpha_0}{x} - \frac{1}{x^2}, -z, -w, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4 \right), \\ s_1 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4 \right), \\ s_2 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(x, y - \frac{\alpha_2 z}{xz - 1}, z, w - \frac{\alpha_2 x}{xz - 1}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 \right), \\ s_3 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3 \right), \\ s_4 : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(x, y, z, w - \frac{2\alpha_4}{z} + \frac{t}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4 \right), \\ \psi : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\longrightarrow \left(\frac{z}{t}, tw, tx, \frac{y}{t}, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0 \right). \end{aligned}$$

Remark We correct the definition of s_4 by Sasano [28], which is given by

$$s_4 : (x, y, z, w, t) \longrightarrow \left(x, y, z, w - \frac{2\alpha_4}{w} + \frac{t}{z^2}, -t \right).$$

B.1 The properties of the Bäcklund transformations

Proposition B.1. (0) For $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that $x \equiv 0$.

(1) If $y \equiv 0$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_1 = 0$ or $t + 2z(zw + \alpha_3) = 0$.

(2) If $xz \equiv 1$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_2 = 0$.

(3) If $w \equiv 0$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, then $\alpha_3 = 0$ or $1 + 2x(xy + \alpha_1) = 0$.

(4) For $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, there exists no solution such that $z \equiv 0$.

By Proposition B.1, we have to consider the infinite solution such that $x \equiv \infty$, or $z \equiv \infty$. For this purpose, we first treat the case where $y \equiv 0$ and $\alpha_1 \neq 0$, and the case where $w \equiv 0$ and $\alpha_3 \neq 0$.

Proposition B.2. Suppose that $y \equiv 0$ and $\alpha_1 \neq 0$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$. It then follows that $\alpha_4 \neq 0$ and either of the following occurs:

(1) $\alpha_0 + \alpha_1 = \alpha_3 + \alpha_4 = 0$ and

$$x = -\frac{1}{2\alpha_1}, y = 0, z = \frac{t}{2\alpha_4}, w = 0,$$

(2) $\alpha_0 = 1/2, \alpha_1 = -1/2$ and

$$x = 1 + \frac{4\alpha_4(\alpha_3 + \alpha_4)}{t}, y = 0, z = \frac{t}{2\alpha_4}, w = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}.$$

Proposition B.2 implies that we have to consider the infinite solution such that $x \equiv \infty$.

Proposition B.3. Suppose that $w \equiv 0$ and $\alpha_3 \neq 0$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$. It then follows that $\alpha_0 \neq 0$, and either of the following occurs:

(1) $\alpha_0 + \alpha_1 = \alpha_3 + \alpha_4 = 0$ and

$$x = \frac{1}{2\alpha_0}, y = 0, z = -\frac{t}{2\alpha_3}, w = 0,$$

(2) $\alpha_3 = -1/2, \alpha_4 = 1/2$ and

$$x = \frac{1}{2\alpha_0}, y = -2\alpha_0(\alpha_0 + \alpha_1), z = t + 4\alpha_0(\alpha_0 + \alpha_1), w = 0.$$

Proposition B.3 implies that we have to consider the infinite solution such that $z \equiv \infty$.

B.2 The infinite solution

B.2.1 The case where $x \equiv \infty$

In order to determine the solution such that $x \equiv \infty$, following Sasano [28], we introduce the coordinate transformation r_1 , which is given by

$$r_1 : \quad x_1 = 1/x, \quad y_1 = -(xy + \alpha_1)x, \quad z_1 = z, \quad w_1 = w.$$

Proposition B.4. *Suppose that $x \equiv \infty$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$. It then follows that $\alpha_4 \neq 0$ and either of the following occurs:*

(1) $\alpha_0 = \alpha_3 + \alpha_4 = 0$

$$x_1 = 0, \quad y_1 = \frac{1}{2}, \quad z_1 = \frac{t}{2\alpha_4}, \quad w_1 = 0,$$

that is, $x = \infty, y = 0, z = t/\{2\alpha_4\}, w = 0$,

(2) $\alpha_0 = 0, \alpha_1 = 1/2$ and

$$x_1 = 0, \quad y_1 = \frac{1}{2} + \frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}, \quad z_1 = \frac{t}{2\alpha_4}, \quad w_1 = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t},$$

that is, $x = \infty, y = 0, z = t/\{2\alpha_4\}, w = -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}$.

Proof. For $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, r_1 transforms the system of (x, y, z, w) into the system of (x_1, y_1, z_1, w_1) , which is given by

$$(*) \quad \begin{cases} tx'_1 = 2x_1^2 y_1 + 2\alpha_1 x_1 + t + 2\alpha_0 x_1 - x_1^2 + 2z_1(z_1 w_1 + \alpha_3), \\ ty'_1 = -2x_1 y_1^2 - 2\alpha_0 y_1 + 2x_1 y_1 - 2\alpha_1 y_1 + \alpha_1, \\ tz'_1 = 2z_1^2 w_1 - z_1^2 + (1 - 2\alpha_4)z_1 + t + 2y_1 z_1^2, \\ tw'_1 = -2z_1 w_1^2 + 2z_1 w_1 - (1 - 2\alpha_4)w_1 + \alpha_3 - 2y_1(2z_1 w_1 + \alpha_3). \end{cases}$$

Setting $x_1 \equiv 0$, we have

$$t + 2z_1^2 w_1 + 2\alpha_3 z_1 = 0, \tag{B.1}$$

which implies that

$$1 + 4z_1 w_1 z'_1 + 2z_1^2 w'_1 + 2\alpha_3 z'_1 = 0. \tag{B.2}$$

From $(*)$, (B.1) and (B.2), it then follows that $z_1 w_1 = -\alpha_3 - \alpha_4$. Thus, considering (B.1), we find that $\alpha_4 \neq 0$ and

$$z_1 = \frac{t}{2\alpha_4}, \quad w_1 = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}. \tag{B.3}$$

Substituting (B.3) into

$$tz'_1 = 2z_1^2 w_1 - z_1^2 + (1 - 2\alpha_4)z_1 + t + 2y_1 z_1^2,$$

we obtain

$$y_1 = \frac{1}{2} + \frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}. \quad (\text{B.4})$$

Moreover, substituting (B.3), (B.4) and $x_1 \equiv 0$ into

$$ty'_1 = -2x_1 y_1^2 - 2\alpha_0 y_1 + 2x_1 y_1 - 2\alpha_1 y_1 + \alpha_1,$$

we have

$$\{1 - 2(\alpha_0 + \alpha_1)\} \frac{2\alpha_4(\alpha_3 + \alpha_4)}{t} - \alpha_0 = 0, \quad (\alpha_4 \neq 0),$$

which proves the proposition. \square

Remark In both cases of Proposition B.4, we can express the solution by

$$x_1 \equiv 0, y_1 = \frac{1}{2} + \frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}, z_1 = \frac{t}{2\alpha_4}, w_1 = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t},$$

and

$$x = \infty, y = 0, z = \frac{t}{2\alpha_4}, w = -\frac{2\alpha_4(\alpha_3 + \alpha_4)}{t}.$$

B.2.2 The case where $z \equiv \infty$

In order to determine the solution such that $z \equiv \infty$, following Sasano [28], we introduce the coordinate transformation r_3 , which is given by

$$r_3 : \quad x_3 = x, y_3 = y, z_3 = 1/z, w_3 = -z(zw + \alpha_3).$$

Proposition B.5. *Suppose that $z \equiv \infty$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$. It then follows that $\alpha_0 \neq 0$ and either of the following occurs:*

(1) $\alpha_0 + \alpha_1 = \alpha_4 = 0$ and

$$x_3 = \frac{1}{2\alpha_0}, y_3 = 0, z_3 = 0, w_3 = \frac{t}{2},$$

that is, $x = 1/\{2\alpha_0\}, y = 0, z = \infty, w = 0$,

(2) $\alpha_3 = 1/2, \alpha_4 = 0$ and

$$x_3 = \frac{1}{2\alpha_0}, y_3 = -2\alpha_0(\alpha_0 + \alpha_1), z_3 = 0, w_3 = \frac{t}{2} + 2\alpha_0(\alpha_0 + \alpha_1),$$

that is, $x = 1/\{2\alpha_0\}, y = -2\alpha_0(\alpha_0 + \alpha_1), z = \infty, w = 0$.

Proof. For $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, r_3 transforms the system of (x, y, z, w) into the system of (x_1, y_1, z_1, w_1) , which is given by

$$(* * *) \begin{cases} tx'_3 = 2x_3^2 y_3 - tx_3^2 - 2\alpha_0 x_3 + 1 + 2x_3^2 w_3, \\ ty'_3 = -2x_3 y_3^2 + 2tx_3 y_3 + 2\alpha_0 y_3 + \alpha_1 t - 2w_3(2x_3 y_3 + \alpha_1), \\ tz'_3 = 2z_3^2 w_3 + 2\alpha_3 z_3 + 1 - (1 - 2\alpha_4)z_3 - tz_3^2 + 2x_3(x_3 y_3 + \alpha_1), \\ tw'_3 = -2z_3 w_3^2 - 2tz_3 w_3 + (1 - 2\alpha_3 - 2\alpha_4)w_3 + \alpha_3 t. \end{cases}$$

Setting $z_3 \equiv 0$, we have

$$1 + 2x_3^2 y_3 + 2\alpha_1 x_3 = 0, \quad (\text{B.5})$$

which implies that

$$4x_3 y_3 x'_3 + 2x_3^2 y'_3 + 2\alpha_1 x'_3 = 0 \quad (\text{B.6})$$

From $(* * *)$, (B.5) and (B.6), it then follows that $x_3 y_3 = -\alpha_0 - \alpha_1$. Thus, considering (B.5), we find that $\alpha_0 \neq 0$ and

$$x_3 = \frac{1}{2\alpha_0}, \quad y_3 = -2\alpha_0(\alpha_0 + \alpha_1). \quad (\text{B.7})$$

Substituting (B.7) into

$$tx'_3 = 2x_3^2 y_3 - tx_3^2 - 2\alpha_0 x_3 + 1 + 2x_3^2 w_3,$$

we obtain

$$w_3 = \frac{t}{2} + 2\alpha_0(\alpha_0 + \alpha_1). \quad (\text{B.8})$$

Moreover, substituting (B.7), (B.8) and $z_3 \equiv 0$ into

$$tw'_3 = -2z_3 w_3^2 - 2tz_3 w_3 + (1 - 2\alpha_3 - 2\alpha_4)w_3 + \alpha_3 t,$$

we have

$$-\alpha_4 t + 2\alpha_0(\alpha_0 + \alpha_1)(1 - 2\alpha_3 - 2\alpha_4) = 0, \quad (\alpha_0 \neq 0),$$

which proves the proposition. \square

Remark In both cases of Proposition B.5, we can express the solution by

$$x_3 = \frac{1}{2\alpha_0}, \quad y_3 = -2\alpha_0(\alpha_0 + \alpha_1), \quad z_3 = 0, \quad w_3 = \frac{t}{2} + 2\alpha_0(\alpha_0 + \alpha_1),$$

and

$$x = \frac{1}{2\alpha_0}, \quad y = -2\alpha_0(\alpha_0 + \alpha_1), \quad z = \infty, \quad y = 0.$$

B.2.3 The case where $x = z \equiv \infty$

In order to determine the solution such that $x = z = \infty$, we define $r_5 = r_1 r_3 = r_3 r_1$ by

$$r_5 : \quad x_5 = 1/x, \quad y_5 = -(xy + \alpha_1)x, \quad z_5 = 1/z, \quad w_5 = -(zw + \alpha_3)z.$$

Proposition B.6. *Suppose that $x = z \equiv \infty$ for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$. Then, $\alpha_0 = \alpha_4 = 0$ and*

$$x_5 = 0, \quad y_5 = 1/2, \quad z_5 = 0, \quad w_5 = t/2,$$

that is, $x = \infty, y = 0, z = \infty, w = 0$.

Proof. For $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, r_5 then transforms the system of (x, y, z, w) into the system of (x_5, y_5, z_5, w_5) , which is given by

$$\begin{cases} tx'_5 = 2x_5^2 y_5 - x_5^2 + (2\alpha_0 + 2\alpha_1)x_5 + t - 2w_5, \\ ty'_5 = -2x_5 y_5^2 - (2\alpha_0 + 2\alpha_1)y_5 + 2x_5 y_5 + \alpha_1, \\ tz'_5 = 2z_5^2 w_5 - (1 - 2\alpha_3 - 2\alpha_4)z_5 + 1 - tz_5^2 - 2y_5, \\ tw'_5 = -2z_5 w_5^2 + (1 - 2\alpha_3 - 2\alpha_4)w_5 + 2tz_5 w_5 + \alpha_3 t. \end{cases}$$

Setting $x_5 = z_5 \equiv 0$ in

$$\begin{cases} tx'_5 = 2x_5^2 y_5 - x_5^2 + (2\alpha_0 + 2\alpha_1)x_5 + t - 2w_5, \\ tz'_5 = 2z_5^2 w_5 - (1 - 2\alpha_3 - 2\alpha_4)z_5 + 1 - tz_5^2 - 2y_5, \end{cases}$$

we obtain $w_5 = t/2, y_5 = 1/2$.

Substituting $y_5 = 1/2, w_5 = t/2$ into

$$\begin{cases} ty'_5 = -2x_5 y_5^2 - (2\alpha_0 + 2\alpha_1)y_5 + 2x_5 y_5 + \alpha_1, \\ tw'_5 = -2z_5 w_5^2 + (1 - 2\alpha_3 - 2\alpha_4)w_5 + 2tz_5 w_5 + \alpha_3 t, \end{cases}$$

we have $\alpha_0 = \alpha_4 = 0$.

□

B.3 The Bäcklund transformations and the infinite solutions

Proposition B.7. *Suppose that $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$ has an infinite solution such that $x \equiv \infty$. The actions of the Bäcklund transformations are then as follows:*

- (0) $s_0(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}),$
- (1)-(i) if $\alpha_1 \neq 0,$

$$s_1(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (1/\{2\alpha_1\} + 2\alpha_4(\alpha_3 + \alpha_4)/\{\alpha_1 t\}, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}),$$

(1)-(ii) if $\alpha_1 = 0$,

$$s_1(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}),$$

(2) $s_2(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_2 + \alpha_3 + \alpha_4)t^{-1})$,

(3)-(i) if $\alpha_3 + \alpha_4 \neq 0$,

$$s_3(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, t/\{2(\alpha_3 + \alpha_4)\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}),$$

(3)-(ii) if $\alpha_3 + \alpha_4 = 0$,

$$s_3(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, \infty, 0),$$

(4) $s_4(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (\infty, 0, t/\{2(-\alpha_4)\}, -2(-\alpha_4)(\alpha_3 + \alpha_4)t^{-1})$,

(5) $\psi(\infty, 0, t/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4)t^{-1}) = (1/\{2\alpha_4\}, -2\alpha_4(\alpha_3 + \alpha_4), \infty, 0)$.

Proposition B.8. Suppose that $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$ has an infinite solution such that $z \equiv \infty$. The actions of the Bäcklund transformations are then as follows:

(0) $s_0(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (-1/\{2\alpha_0\}, 2\alpha_0(\alpha_0 + \alpha_1), \infty, 0)$,

(1)-(i) if $\alpha_0 + \alpha_1 \neq 0$,

$$s_1(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (1/\{2(\alpha_0 + \alpha_1)\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0),$$

(1)-(ii) if $\alpha_0 + \alpha_1 = 0$,

$$s_1(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (\infty, 0, \infty, 0),$$

(2) $s_2(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1 + \alpha_2), \infty, 0)$,

(3)-(i) if $\alpha_3 \neq 0$,

$$s_3(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), t/\{2\alpha_3\} + 2\alpha_0(\alpha_0 + \alpha_1)/\alpha_3, 0),$$

(3)-(ii) if $\alpha_3 = 0$,

$$s_3(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0),$$

(4) $s_4(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0)$,

(5) $\psi(1/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1), \infty, 0) = (\infty, 0, t/\{2\alpha_0\}, -2\alpha_0(\alpha_0 + \alpha_1)t^{-1})$.

Proposition B.9. Suppose that $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$ has an infinite solution such that $x = z \equiv \infty$. The actions of the Bäcklund transformations are then as follows:

(0) $s_0(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0)$,

(1)-(i) if $\alpha_1 \neq 0$,

$$s_1(\infty, 0, \infty, 0) = (1/\{2\alpha_1\}, 0, \infty, 0),$$

(1)-(ii) if $\alpha_1 = 0$,

$$s_1(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0),$$

(2) $s_2(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0)$,

(3)-(i) if $\alpha_3 \neq 0$,

$$s_3(\infty, 0, \infty, 0) = (\infty, 0, t/\{2\alpha_3\}, 0),$$

(3)-(ii) if $\alpha_3 = 0$,

$$s_3(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0),$$

(4) $s_4(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0)$,

(5) $\psi(\infty, 0, \infty, 0) = (\infty, 0, \infty, 0)$.

B.4 Main theorem for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$

Sasano [28] proved that $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$ is equivalent to $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$.

Proposition B.10. *Suppose that (x, y, z, w) is a solution of $D_4^{(1)}(\alpha_j)_{0 \leq j \leq 4}$, and*

$$\begin{aligned} X &= \frac{1}{x}, Y = -(xy + \alpha_1)x, Z = \frac{1}{z}, W = -(zw + \alpha_3)z, \\ A_0 &= \frac{\alpha_0 - \alpha_1}{2}, A_1 = \alpha_1, A_2 = \alpha_2, A_3 = \alpha_3, A_4 = \frac{\alpha_4 - \alpha_3}{2}. \end{aligned}$$

(X, Y, Z, W) is then a solution of $D_5^{(2)}(A_j)_{0 \leq j \leq 4}$.

By Theorem A.3 and Proposition B.10, we obtain the following theorem:

Theorem B.11. *Suppose that for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters and solution can then be transformed so that*

$$\alpha_0 = \alpha_3 + \alpha_4 = 0, \alpha_1, \alpha_4 \neq 0 \text{ and } (x, y, z, w) = (\infty, 0, t/\{2\alpha_4\}, 0).$$

Moreover, $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$ has a rational solution if and only if one of the following occurs:

- | | | | | |
|-----|---|---|--|--------|
| (1) | $2\alpha_0 \in \mathbb{Z},$ | $2\alpha_3 + 2\alpha_4 \in \mathbb{Z},$ | $2\alpha_0 \equiv 2\alpha_3 + 2\alpha_4$ | mod 2, |
| (2) | $2\alpha_0 \in \mathbb{Z},$ | $2\alpha_4 \in \mathbb{Z},$ | $2\alpha_0 \equiv 2\alpha_4$ | mod 2, |
| (3) | $2\alpha_0 + 2\alpha_1 \in \mathbb{Z},$ | $2\alpha_3 + 2\alpha_4 \in \mathbb{Z},$ | $2\alpha_0 + 2\alpha_1 \equiv 2\alpha_3 + 2\alpha_4$ | mod 2, |
| (4) | $2\alpha_0 + 2\alpha_1 \in \mathbb{Z},$ | $2\alpha_4 \in \mathbb{Z},$ | $2\alpha_0 + 2\alpha_1 \equiv 2\alpha_4$ | mod 2, |
| (5) | $2\alpha_0 \in \mathbb{Z},$ | $2\alpha_1 \in \mathbb{Z},$ | $2\alpha_1 \equiv 1$ | mod 2, |
| (6) | $2\alpha_3 \in \mathbb{Z},$ | $2\alpha_4 \in \mathbb{Z},$ | $2\alpha_3 \equiv 1$ | mod 2. |

By s_1 , we can obtain the following corollary:

Corollary B.12. *Suppose that for $D_5^{(2)}(\alpha_j)_{0 \leq j \leq 4}$, there exists a rational solution. By some Bäcklund transformations, the parameters and solution can then be transformed so that*

$$\alpha_0 + \alpha_1 = \alpha_3 + \alpha_4 = 0, \alpha_0, \alpha_4 \neq 0 \text{ and } (x, y, z, w) = (1/\{2\alpha_0\}, 0, t/\{2\alpha_4\}, 0).$$

References

- [1] K. Fuji and T. Suzuki, Higher order Painleve system of type $D_{2n+2}^{(1)}$ arising from integrable hierarchy. *Int. Math. Res. Not.* **1** (2008), Art. ID rnm 129.
- [2] K. Fuji and T. Suzuki, Drinfeld-Sokolov hierarchies of type A and fourth order Painleve systems, *Funkcial. Ekvac.* **53** (2010), 143-167.
- [3] B. Gambier, Sur les equations différentiels du second ordre et du premier degre dont l'intégrale est á points critiques fixes, *Acta Math.* **33** (1909), 1-55.
- [4] V. I. Gromak, Algebraic solutions of the third Painlevé equation (Russian), *Dokl. Akad. Nauk BSSR* **23** (1979), 499-502.
- [5] V. I. Gromak, Reducibility of the Painlevé equations, *Differ. Equ.* **20** (1984), 1191-1198.
- [6] A. V. Kitaev, C. K. Law and J. B. McLeod, Rational solutions of the fifth Painlevé equation, *Diff. Integral Eqns.* **7** (1994), 967-1000.
- [7] K. Matsuda, Rational solutions of the A_4 Painlevé equation, *Proc. Japan Acad. A* **81** (2005), 85-88.
- [8] K. Matsuda, Rational solutions of the Noumi and Yamada system of type $A_5^{(1)}$, arXiv:0708.2960.
- [9] K. Matsuda, Rational solutions of the Sasano system of type $A_5^{(2)}$, *SIGMA* **7** (2011), 030, 20 pp.
- [10] K. Matsuda, Rational solutions of the Sasano system of type $A_4^{(2)}$, *J. Phys. A* **44** (2011), 405201-405221.
- [11] K. Matsuda, Rational solutions of the Sasano system of type $A_1^{(1)}$, to appear in *Hokkaido Math. J.*

- [12] K. Matsuda, Rational solutions of the Sasano system of type $D_3^{(2)}$, *to appear in Kyushu J. Math.*
- [13] M. Mazzoco, Rational solutions of the Painlevé VI equation, *J. Phys. A* **34** (2001), 2281-2294.
- [14] T. Miwa, Painleve property of monodromy preserving deformation equations and the analyticity of τ functions, *Publ. Res. Inst. Math. Sci.* **17** no. 2, (1981) 703-721.
- [15] Y. Murata, Rational solutions of the second and the fourth Painlevé equations, *Funkcial. Ekvac.* **28** (1985), 1-32.
- [16] Y. Murata, Classical solutions of the third Painlevé equation, *Nagoya Math. J.* **139** (1995), 37-65.
- [17] M. Noumi and Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Comm. Math. Phys.* **199** (1998), 281-295.
- [18] M. Noumi and Y. Yamada, Higher order Painlevé equations of type $A_l^{(1)}$, *Funkcial. Ekvac.* **41** (1998), 483-503.
- [19] T. Oshima, Classification of Fuchsian systems and their connection problem, *preprint, University of Tokyo, Mathematical Sciences* (2008).
- [20] K. Okamoto, Studies on the Painleve equations. III. Second and fourth Painleve equations, P_{II} and P_{IV} , *Math. Ann.* **275** (1986), 221-255.
- [21] K. Okamoto, Studies on the Painlevé equations. I. Sixth Painleve equation P_{VI} , *Ann. Mat. Pure Appl.* **146** (1987), 337-381.
- [22] K. Okamoto, Studies on the Painleve equations. II. Fifth Painleve equation P_V , *Japan J. Math. (N.S.)* **13** (1987), 47-76.
- [23] K. Okamoto, Studies on the Painleve equations. IV. Third Painleve equation P_{III} , *Funkcial. Ekvac.* **30** (1987), 305-332.
- [24] P. Painlevé, Sur les équations différentiels du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Math.* **25** (1902), 1-85.
- [25] H. Sakai, Isomonodromic deformation and 4-dimensional Painlevé type equations, *preprint, University of Tokyo, Mathematical Sciences* (2010).
- [26] Y. Sasano, Higher order Painlevé equations of type $D_l^{(1)}$, *RIMS kokyuroku* **1473** (2006), 143-163.

- [27] Y. Sasano, Coupled Painleve VI systems in dimension four with affine Weyl group symmetry of type $D_6^{(1)}$ II, *RIMS Kokyuroku Bessatsu* **B5** (2008), 137-152.
- [28] Y. Sasano, Coupled Painleve III systems with affine Weyl group symmetry of types $B_4^{(1)}$, $D_4^{(1)}$ and $D_5^{(2)}$, arXiv:0704.2476
- [29] Y. Sasano, Coupled Painleve systems in dimension four with affine Weyl group symmetry of types $A_4^{(2)}$ and $A_1^{(1)}$, arXiv:0809.2399
- [30] Y. Sasano, Coupled Painleve systems with affine Weyl group symmetry of types $D_3^{(2)}$ and $D_5^{(2)}$, arXiv:0705.4540
- [31] Y. Sasano, Symmetries in the system of type $A_5^{(2)}$, arXiv:0704.2327
- [32] Y. Sasano, Four-dimensional Painlevé systems of types $D_5^{(1)}$ and $B_4^{(1)}$, arXiv:0704.3386
- [33] A. P. Vorob'ev, On rational solutions of the second Painlevé equation, *Differ. Equ.* **1** (1965), 58-59.
- [34] A. I. Yablonskii, On rational solutions of the second Painlevé equation (Russian), *Vesti. A. N. BSSR, Ser. Fiz-Tekh. Nauk* **3** (1959), 30-35.
- [35] W. Yuang and Y. Li, Rational solutions of Painleve equations, *Canad. J. Math.* **54** (2002), 648-670.